Extremely Deep Proofs

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- Several other size/space tradeoffs for various proof systems [R17,BN20,R18]

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- Resolution — Focus on for today
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Given an unsatisfiable CNF formula F as a set of clauses

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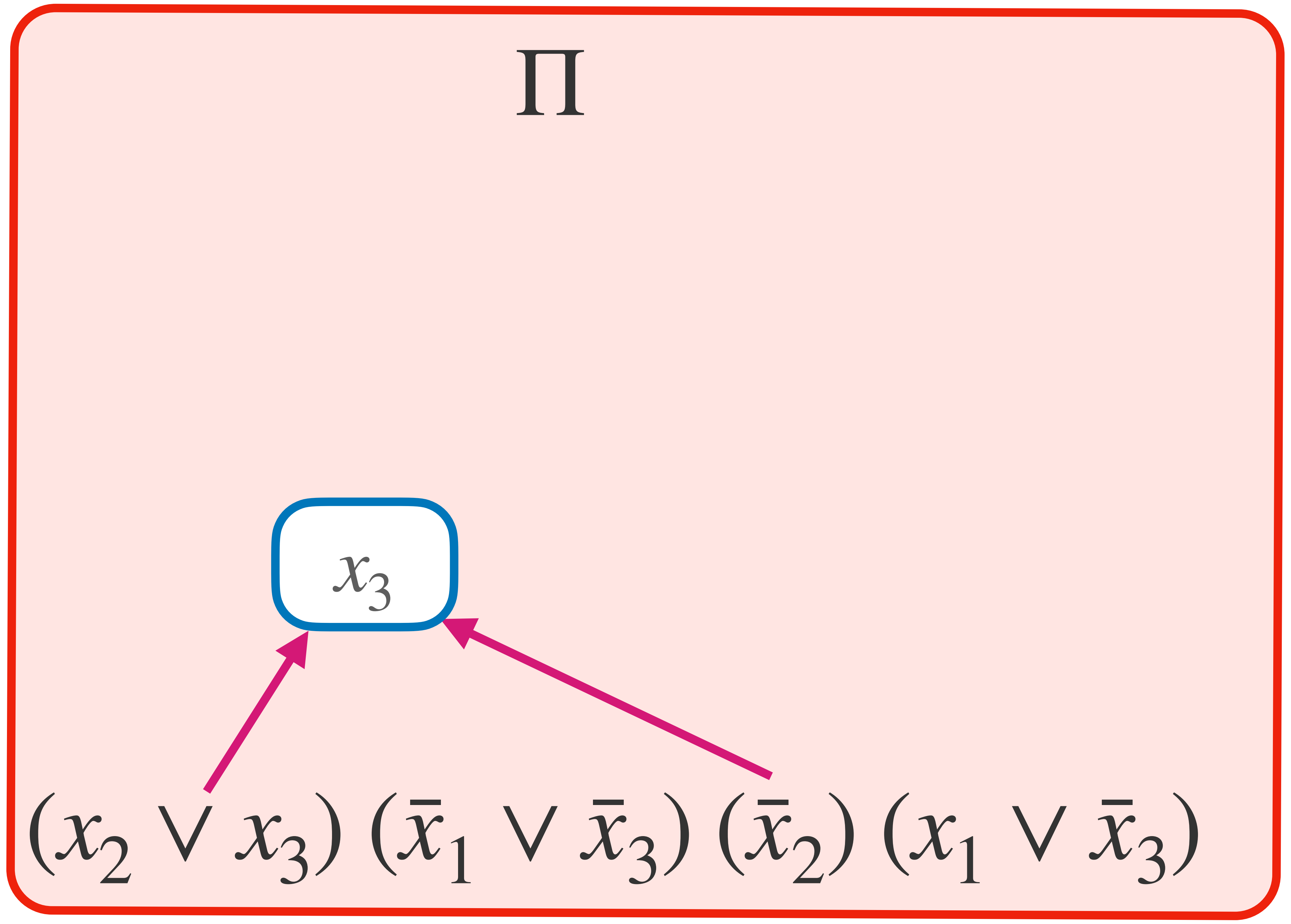

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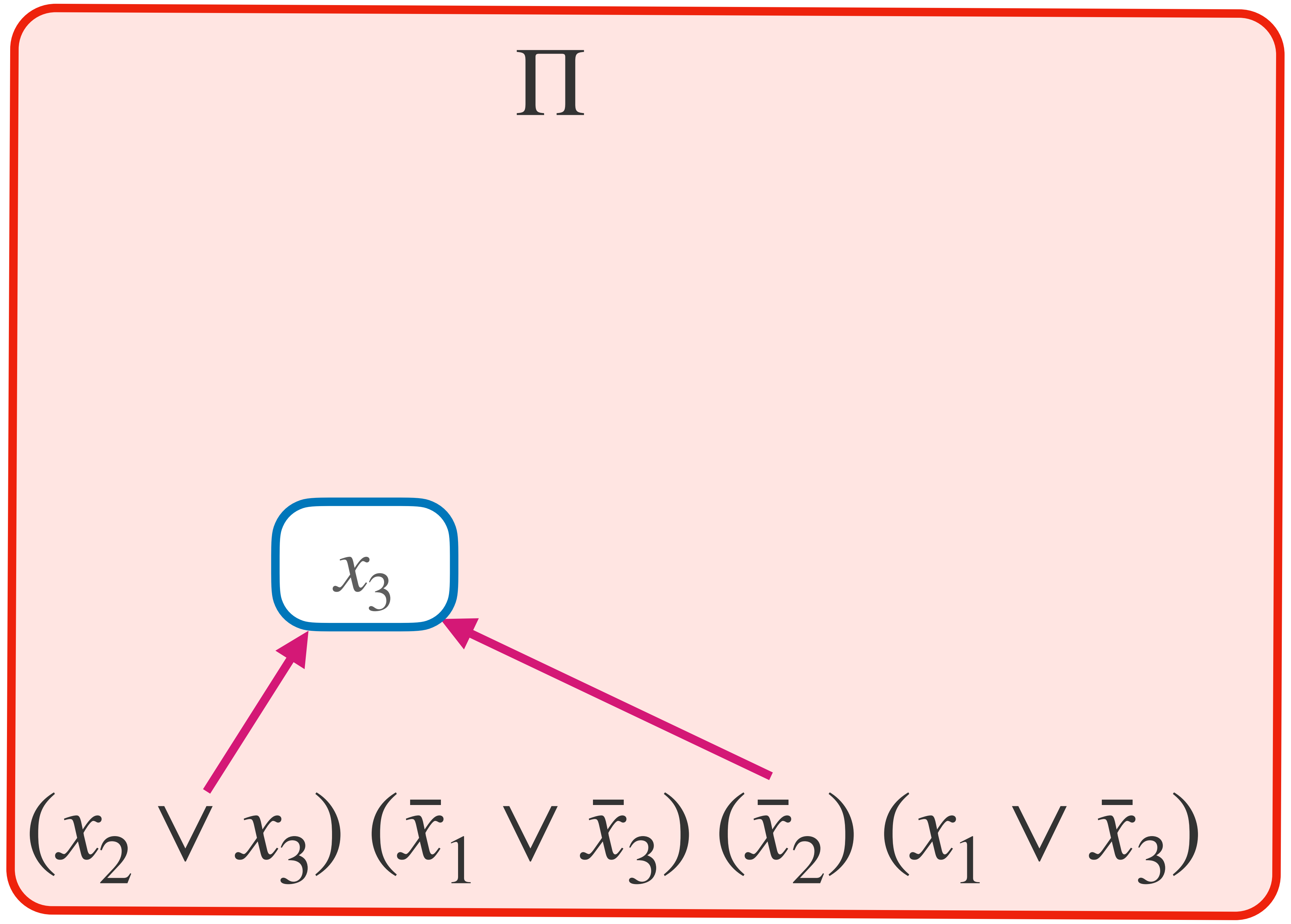

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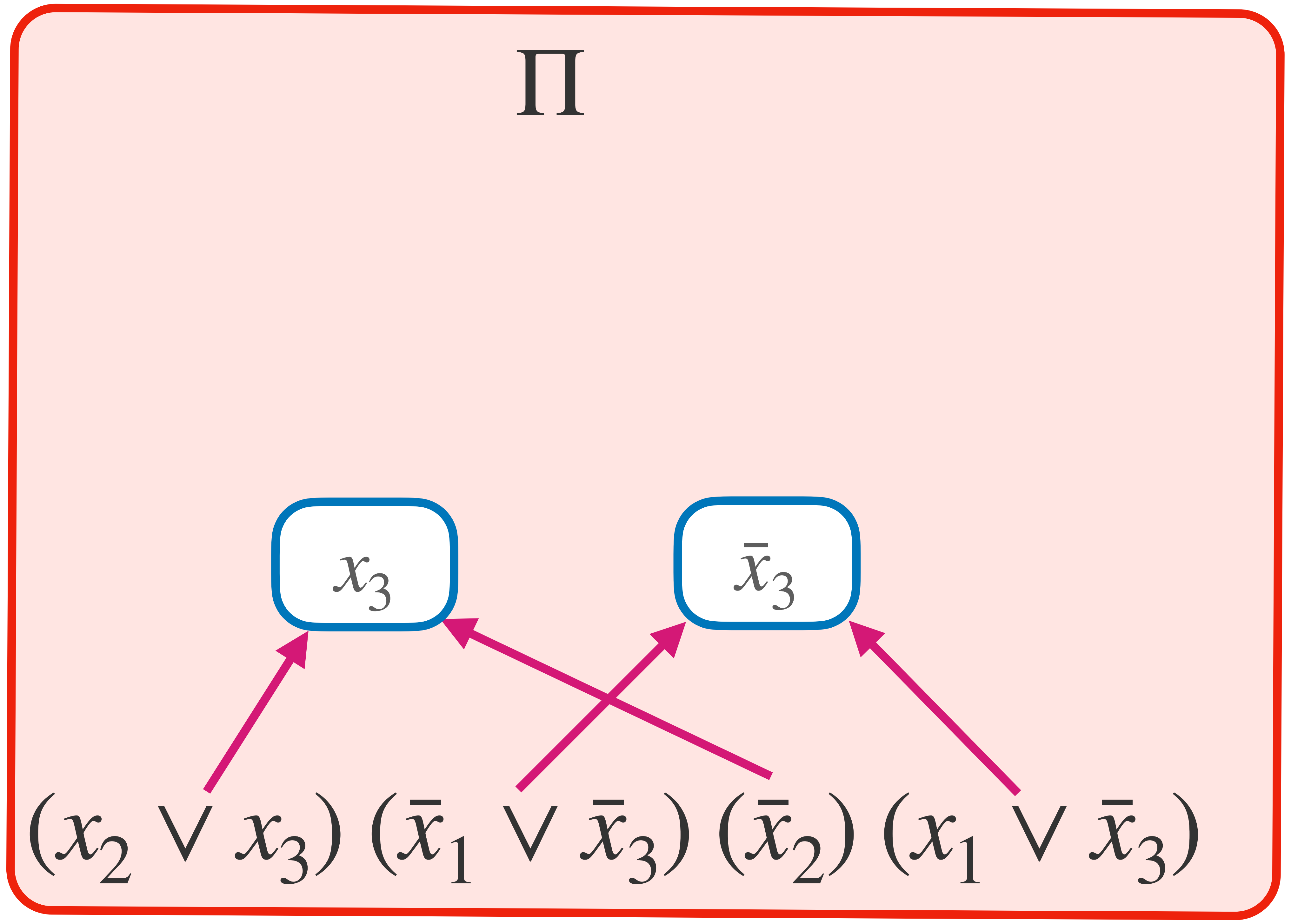

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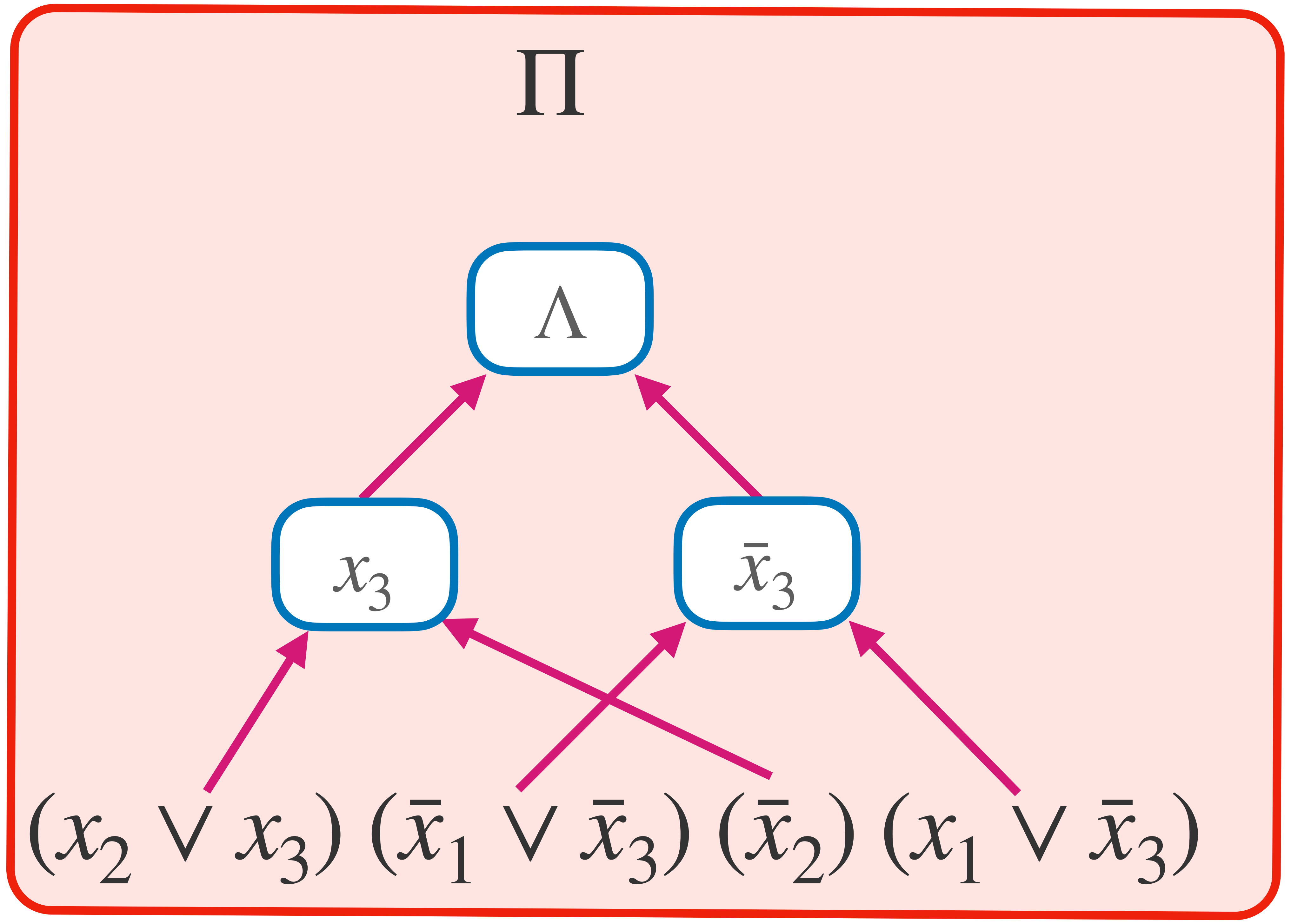

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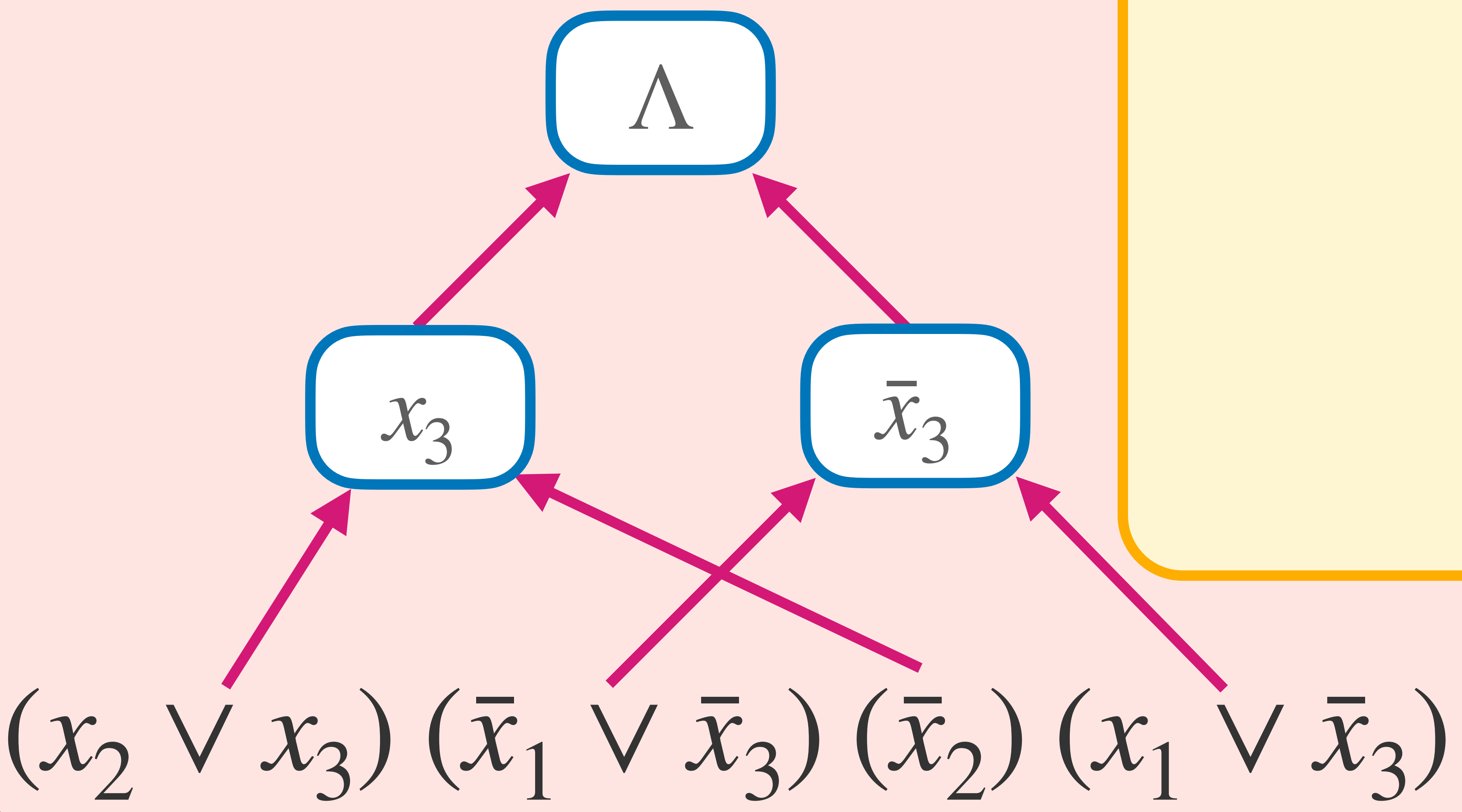

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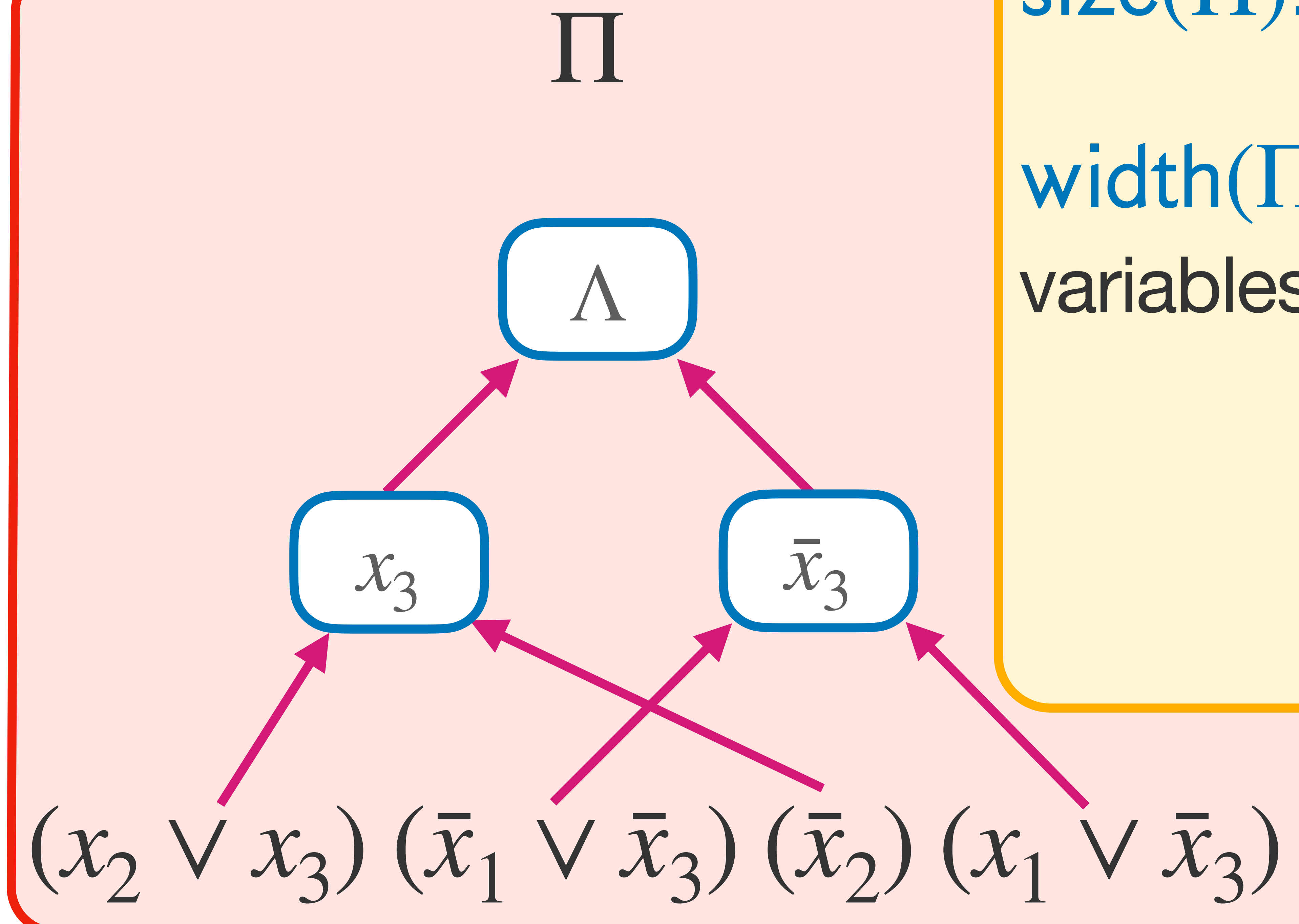

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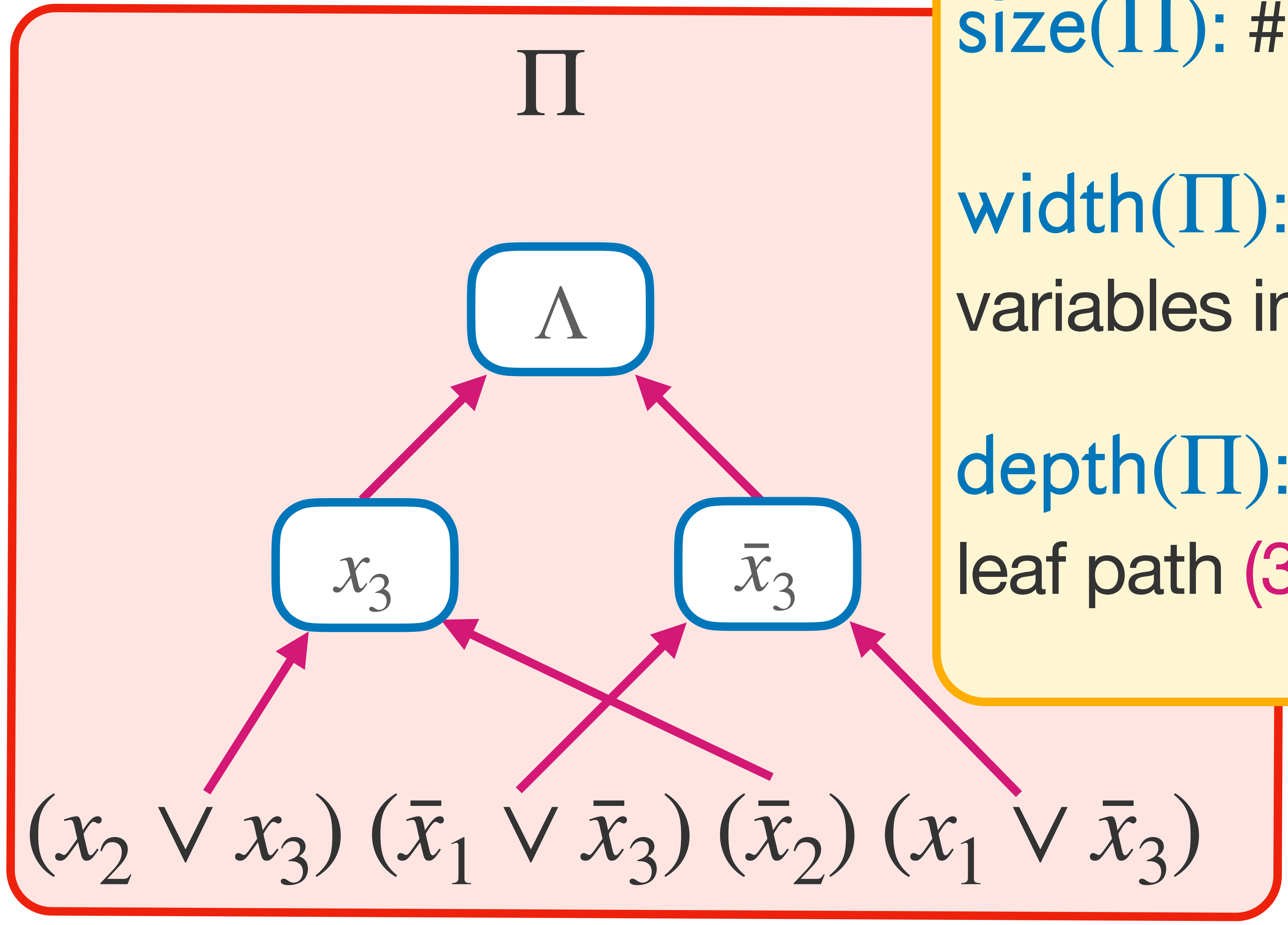

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depth(Π): longest root-to-leaf path (3)

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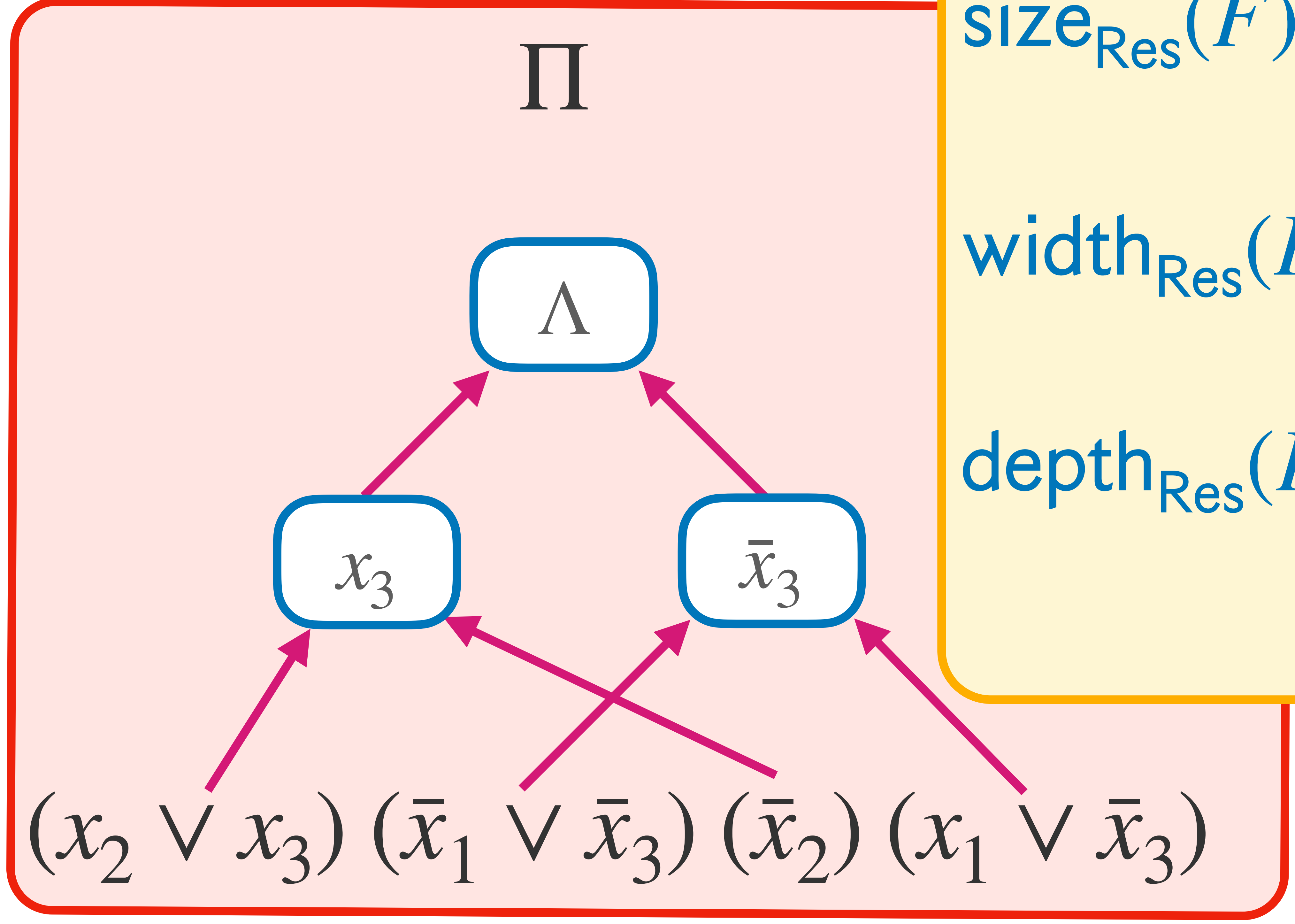

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$\mathrm{size}_{\mathrm{Res}}(F) = \min_{\Pi} \mathrm{size}(\Pi)$

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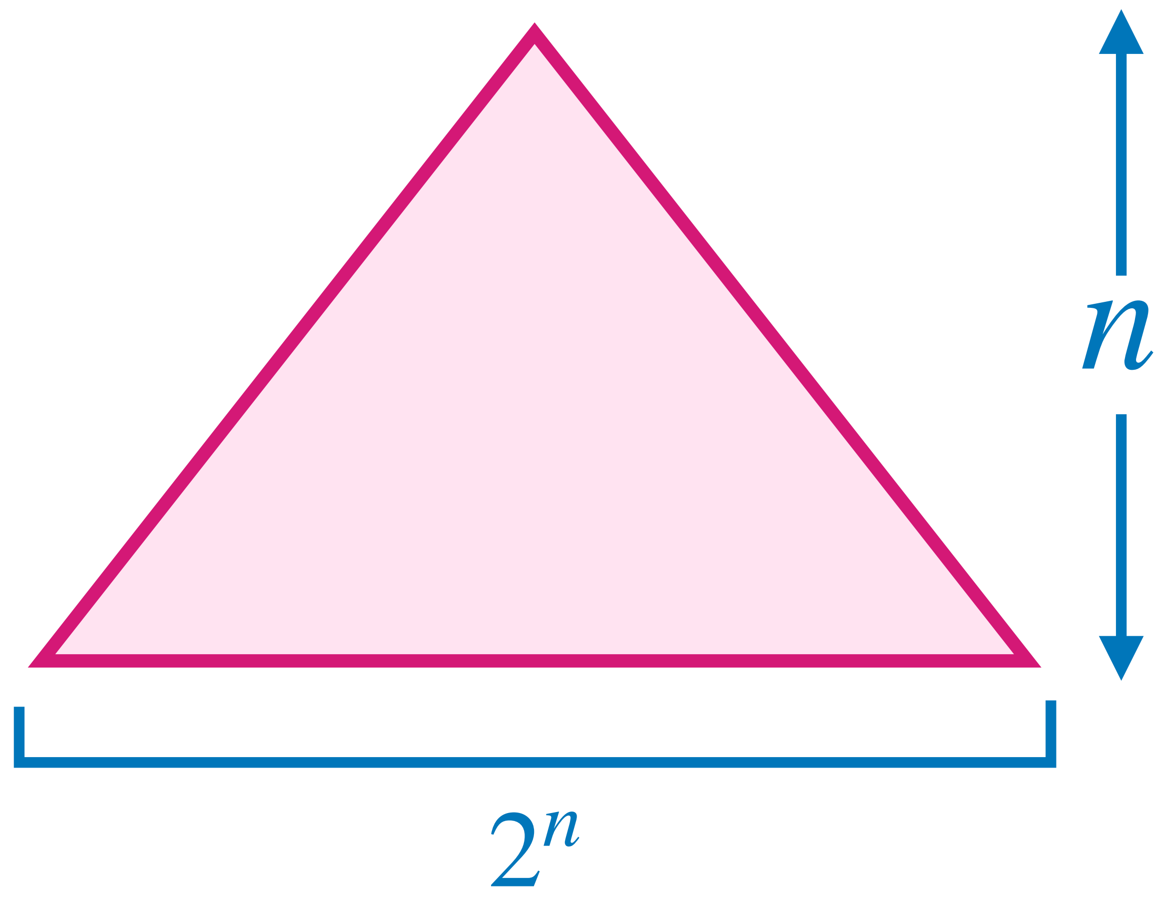

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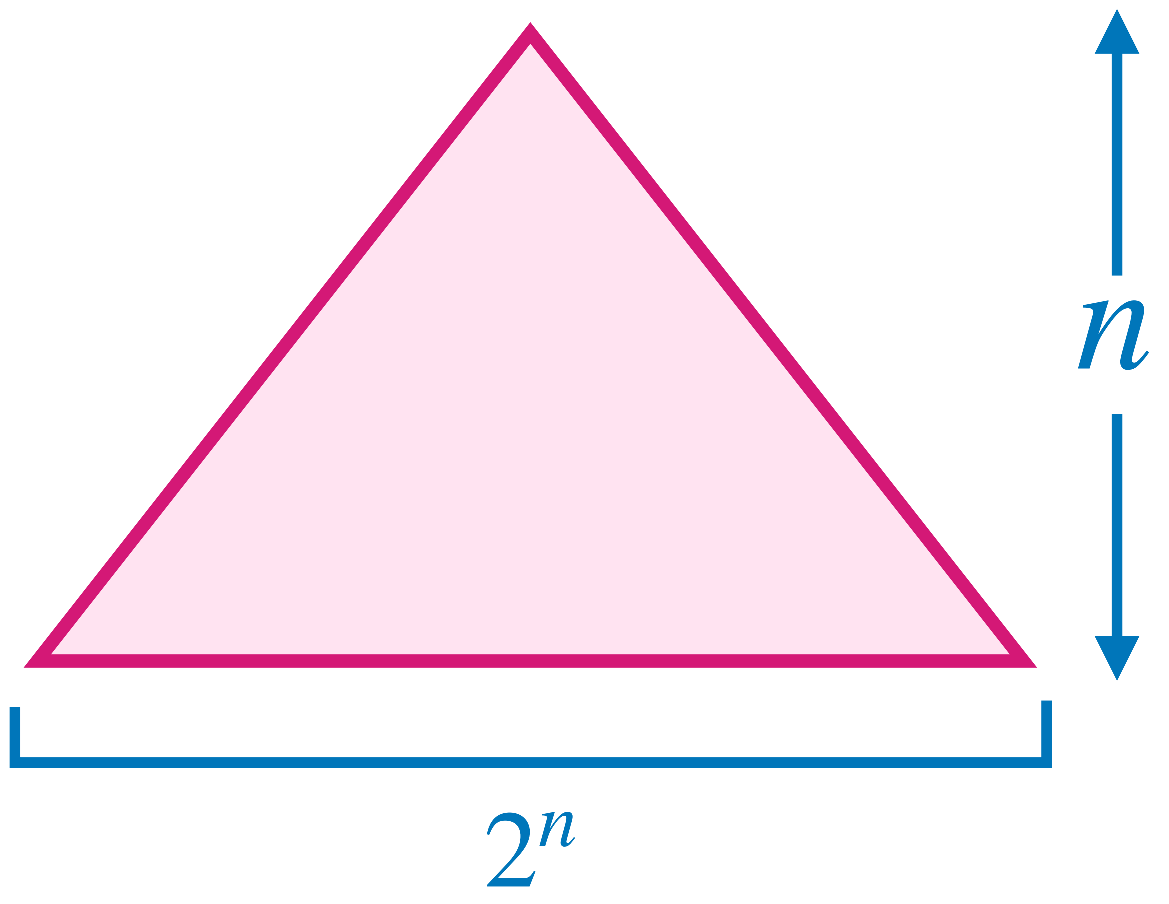

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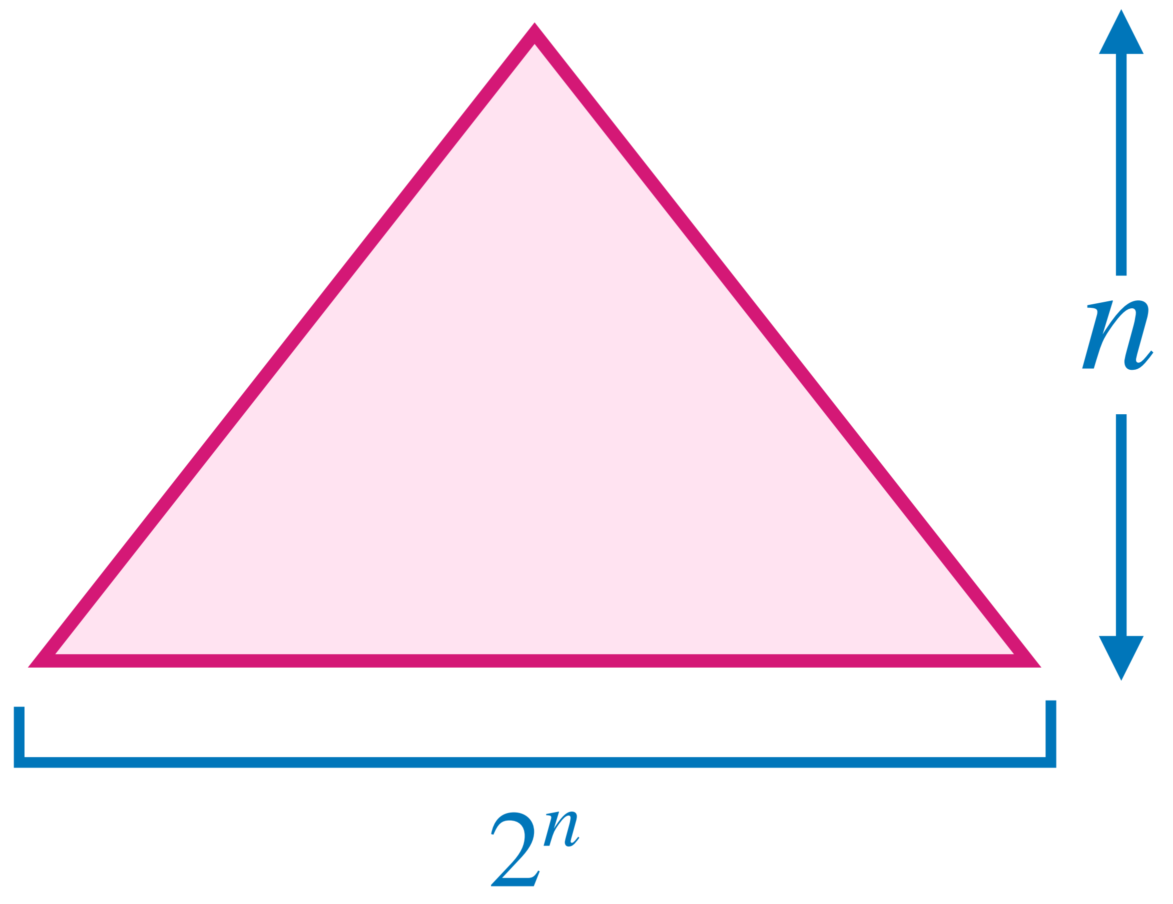

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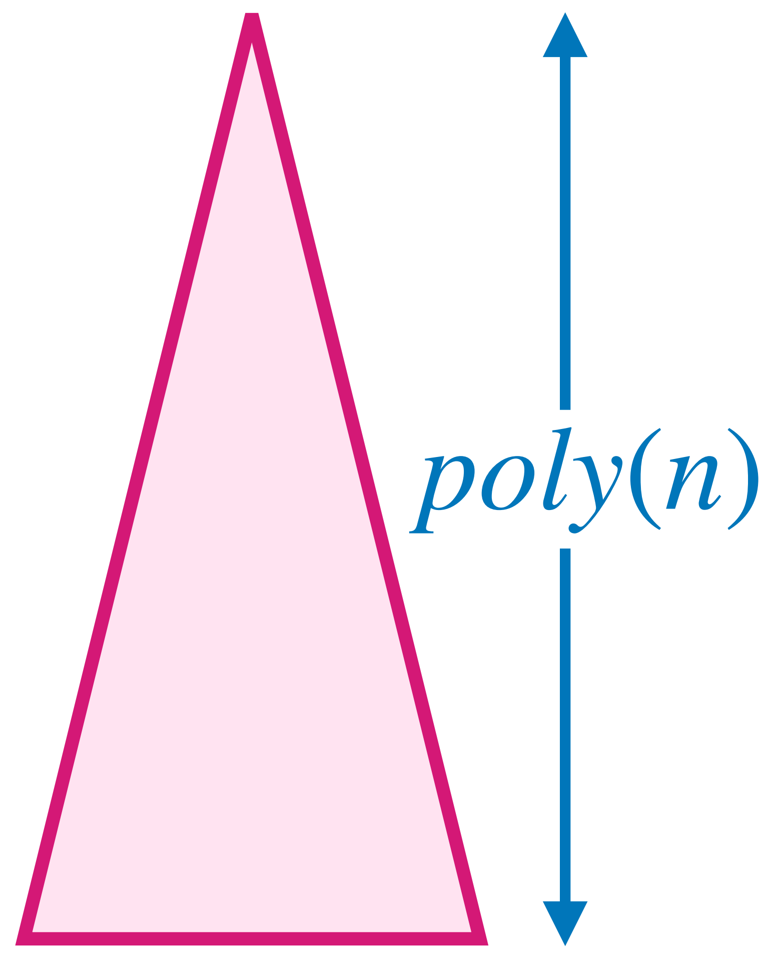
→ Resolution (Res(k), Cutting Planes) cannot always be balanced

This Work

For any $P \in \{$Resolution, Res(k), Cutting Planes$\}$

There is a CNF formula F on n variables such that

- There is a polynomial size P-proof of F
- Any subexponential-size P-proof of F must have poly$(n) > n$ depth



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For any $P \in \{$Resolution, Res(k), Cutting Planes$\}$

There is a CNF formula F on n variables such that

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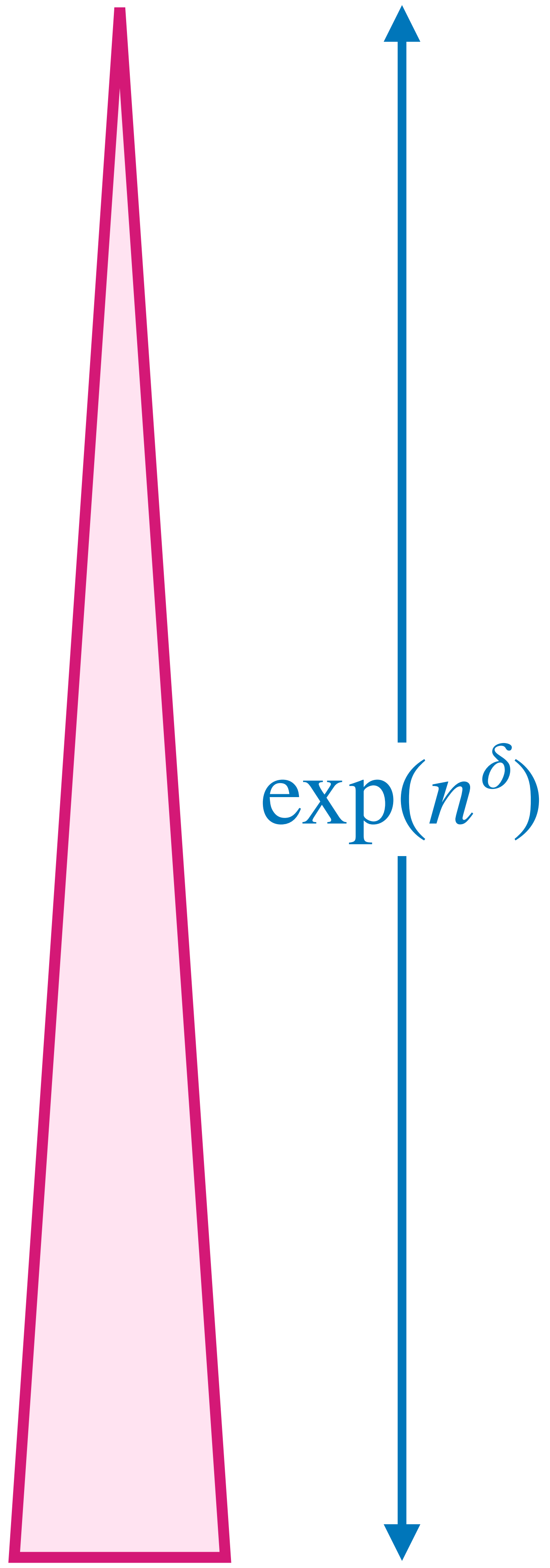
- Any subexponential-size P-proof of F must have weakly exponential depth



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* Caveat: F has $n^{O(c)}$ many clauses — we'll come back to this!

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How do we do compression? Lifting!

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Let P, Q be two proof systems

A lifting theorem relates the complexity of

- P-proofs of F
- Q-proofs of $F \circ g$

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$\implies$ Naively simulation is essentially the best! (A theme of lifting theorems)

Lifting (Composition)

Typically

- P is a “weak” proof system
- Q is a “strong” proof system

A lifting theorem shows that the most efficient Q-proof of $F \circ g$ is to simulate the most efficient P-proof of F (with some extra overhead to handle g)

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Find a gadget g such that

1. The number of variables n of $F \circ g$ will be much smaller than N
2. Any small-size Resolution proof of $F \circ g$ will require the same depth as proving F

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$\rightarrow$ Composing will reduce the total number of variables to $n \ll N$

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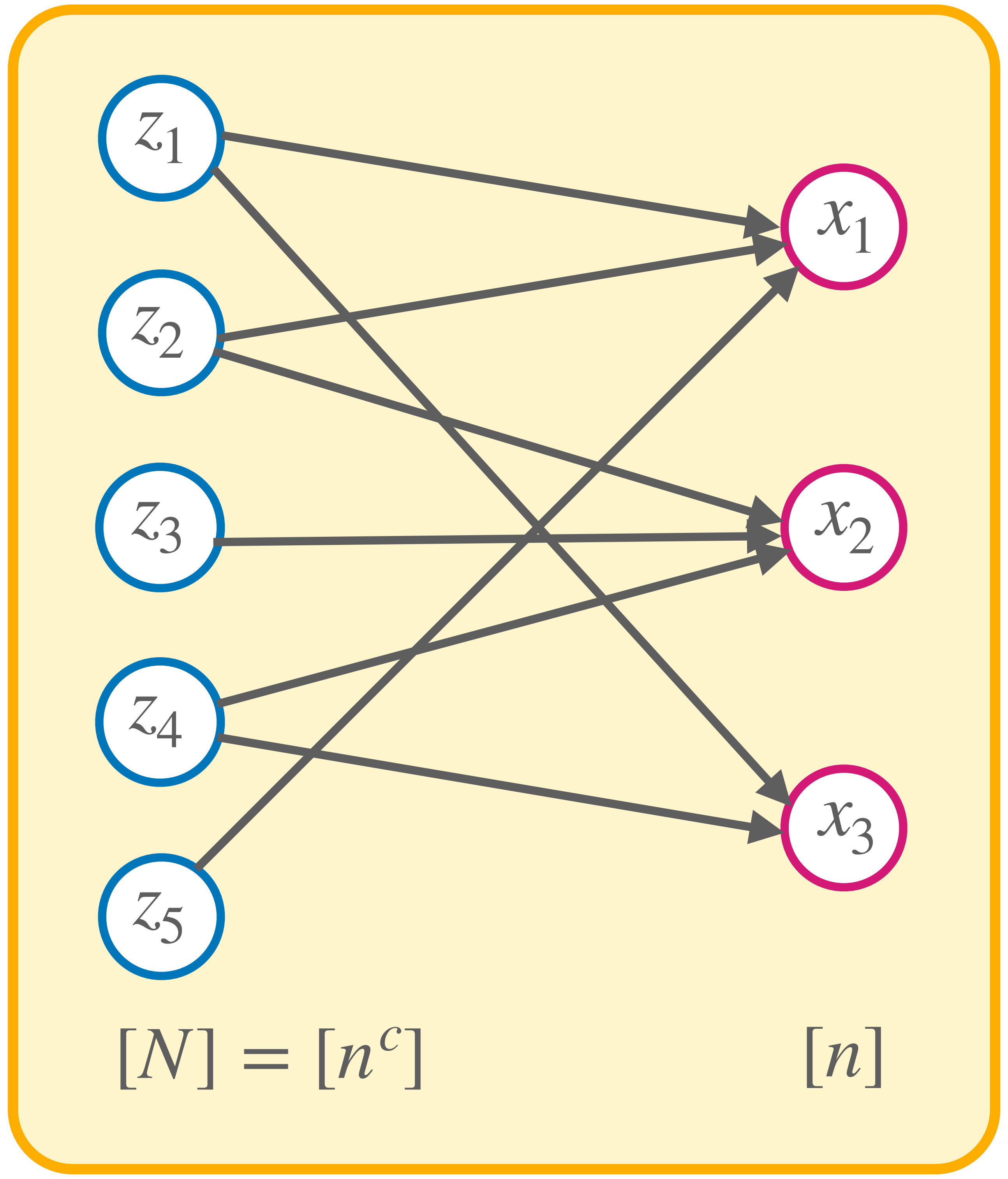

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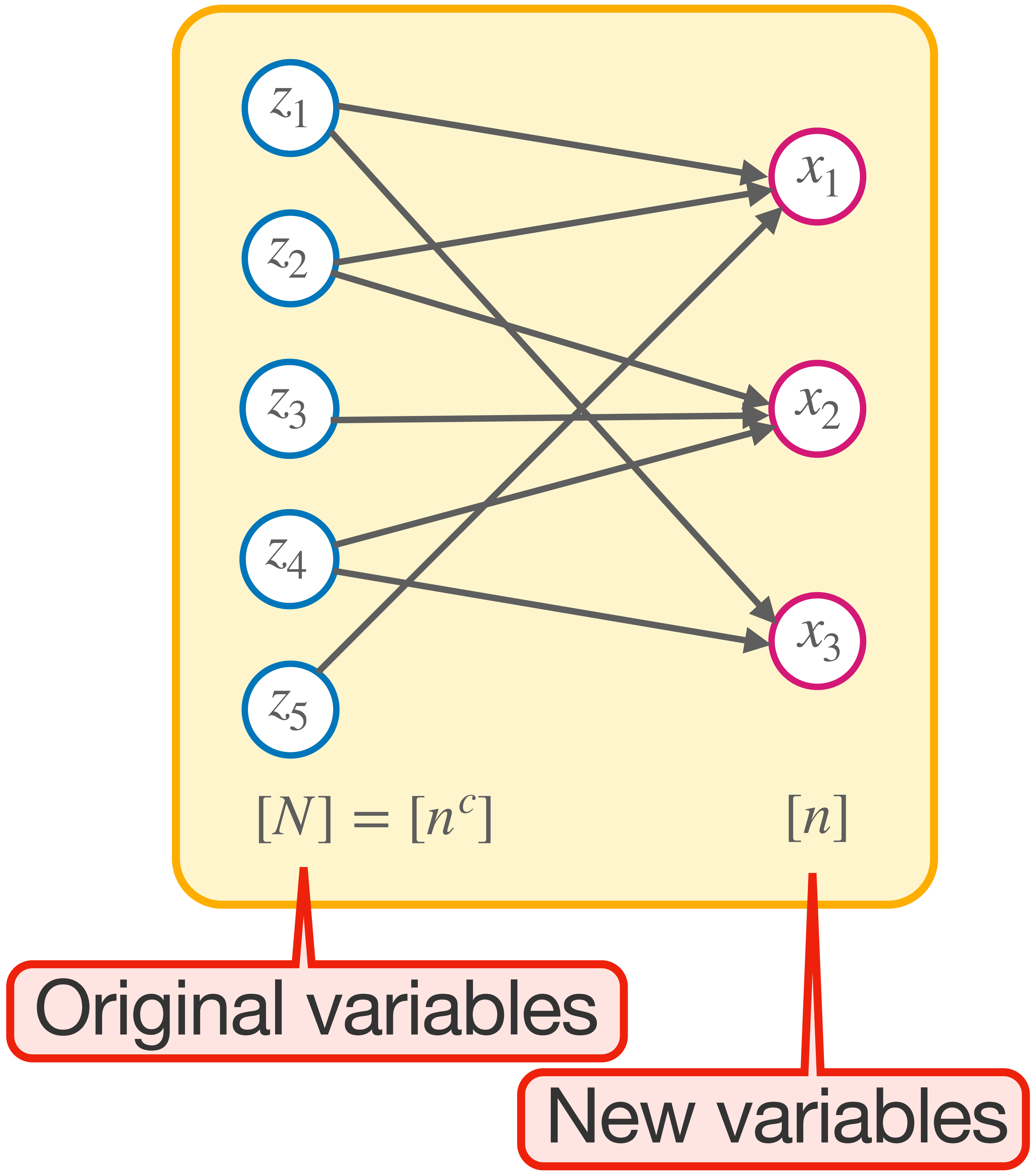

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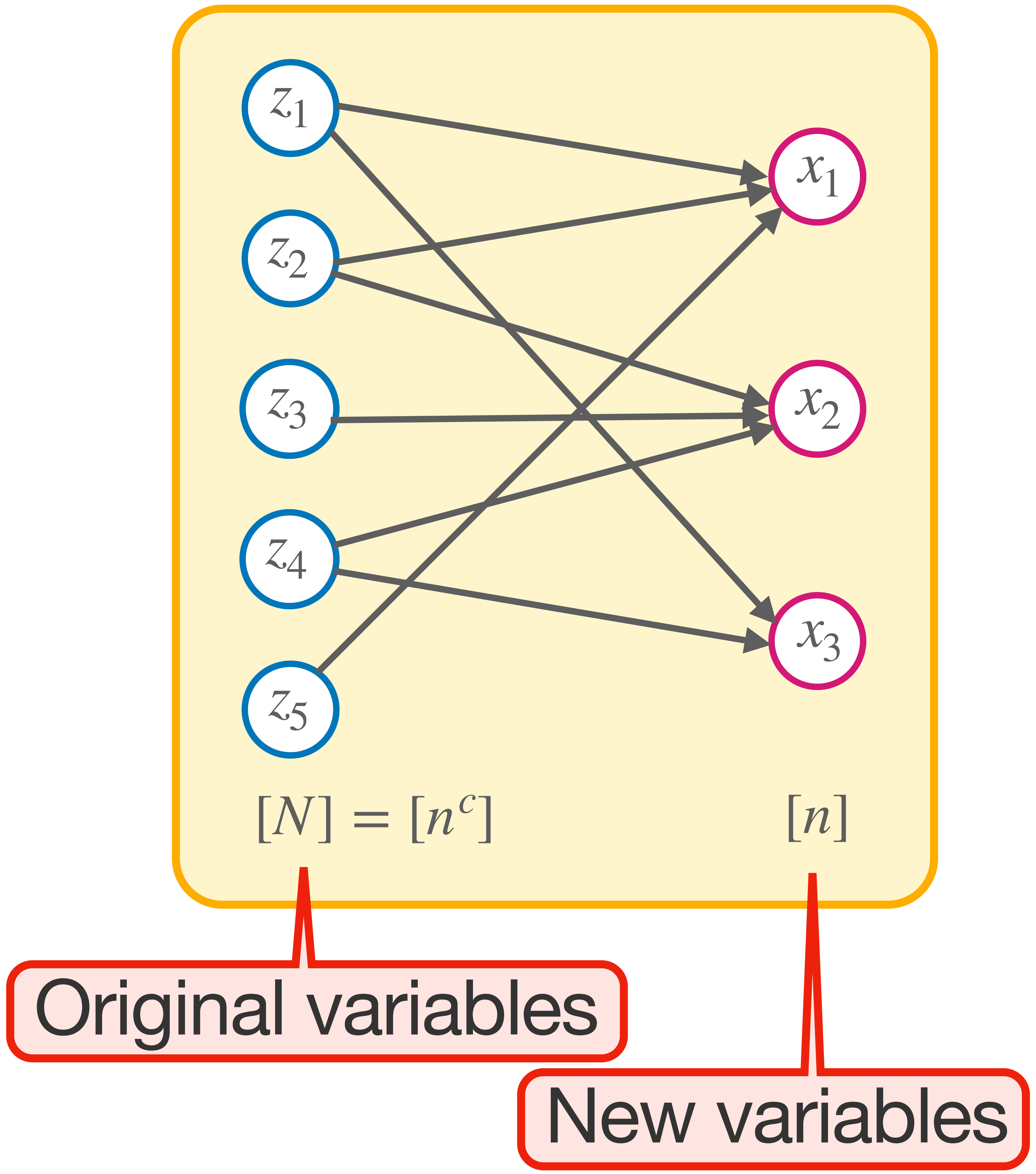
$F \circ \mathsf{XOR}_G$ replaces $z_i \mapsto \bigoplus_{x_j \in \mathsf{N}(z_i)} x_j$



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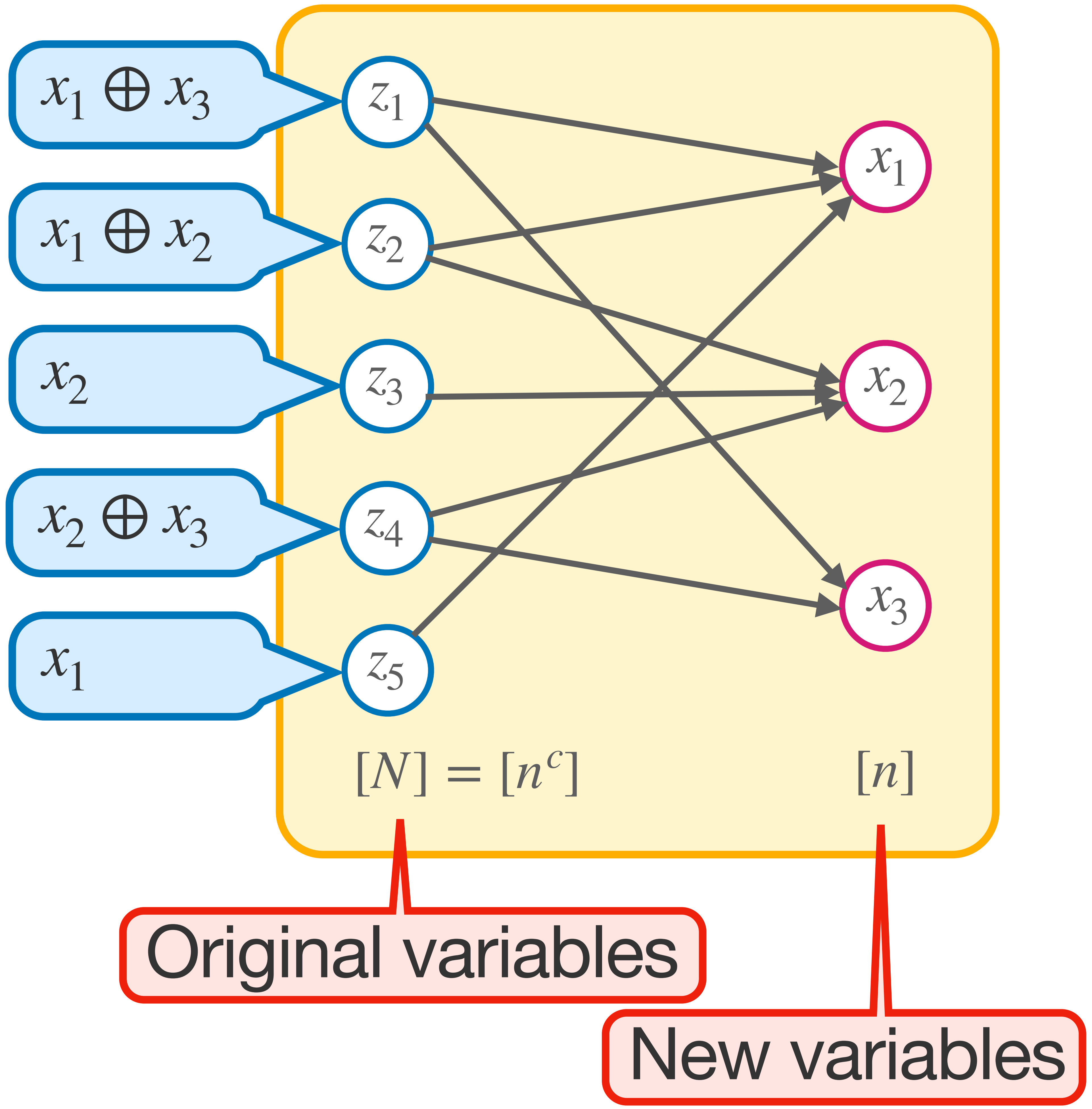

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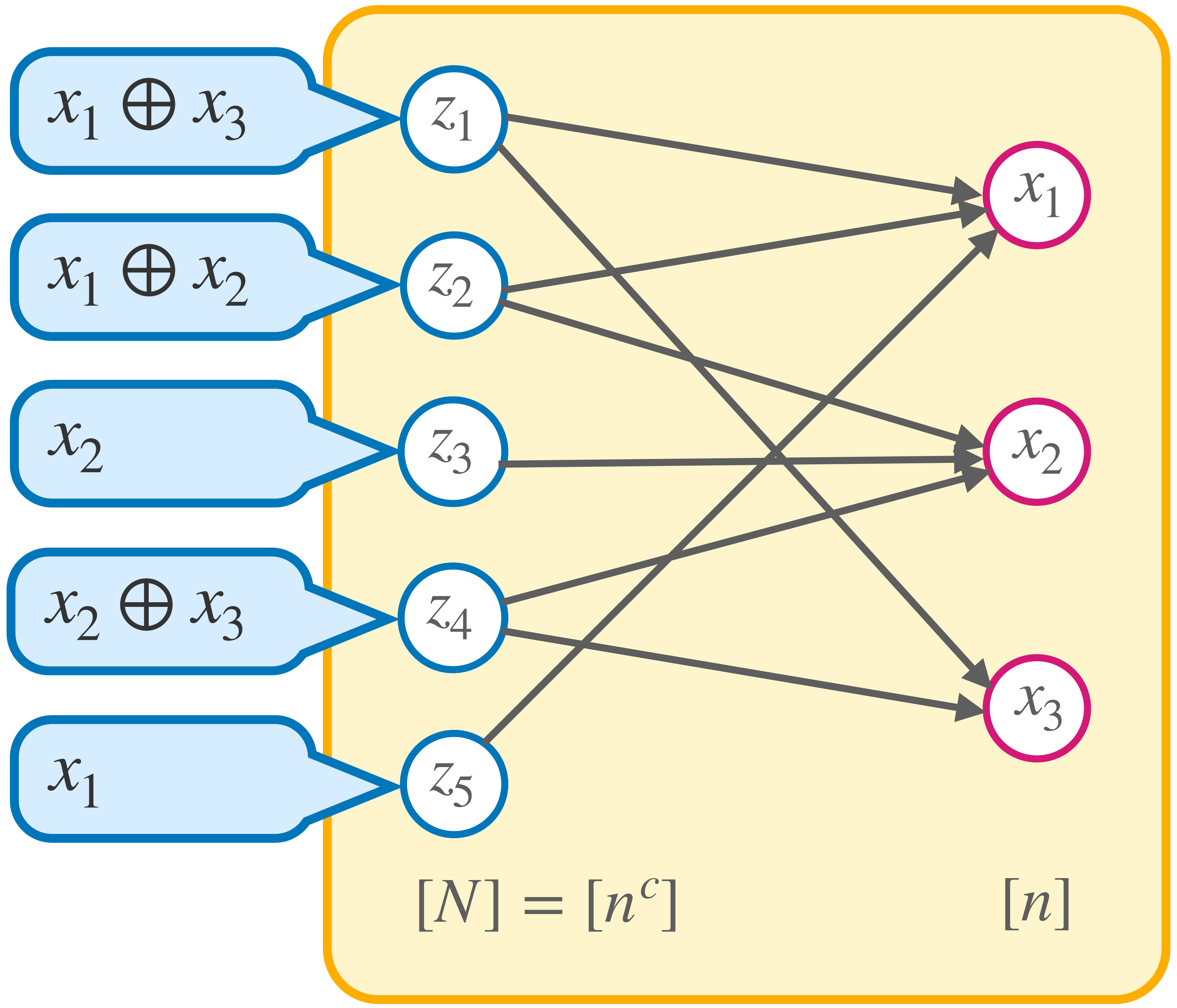

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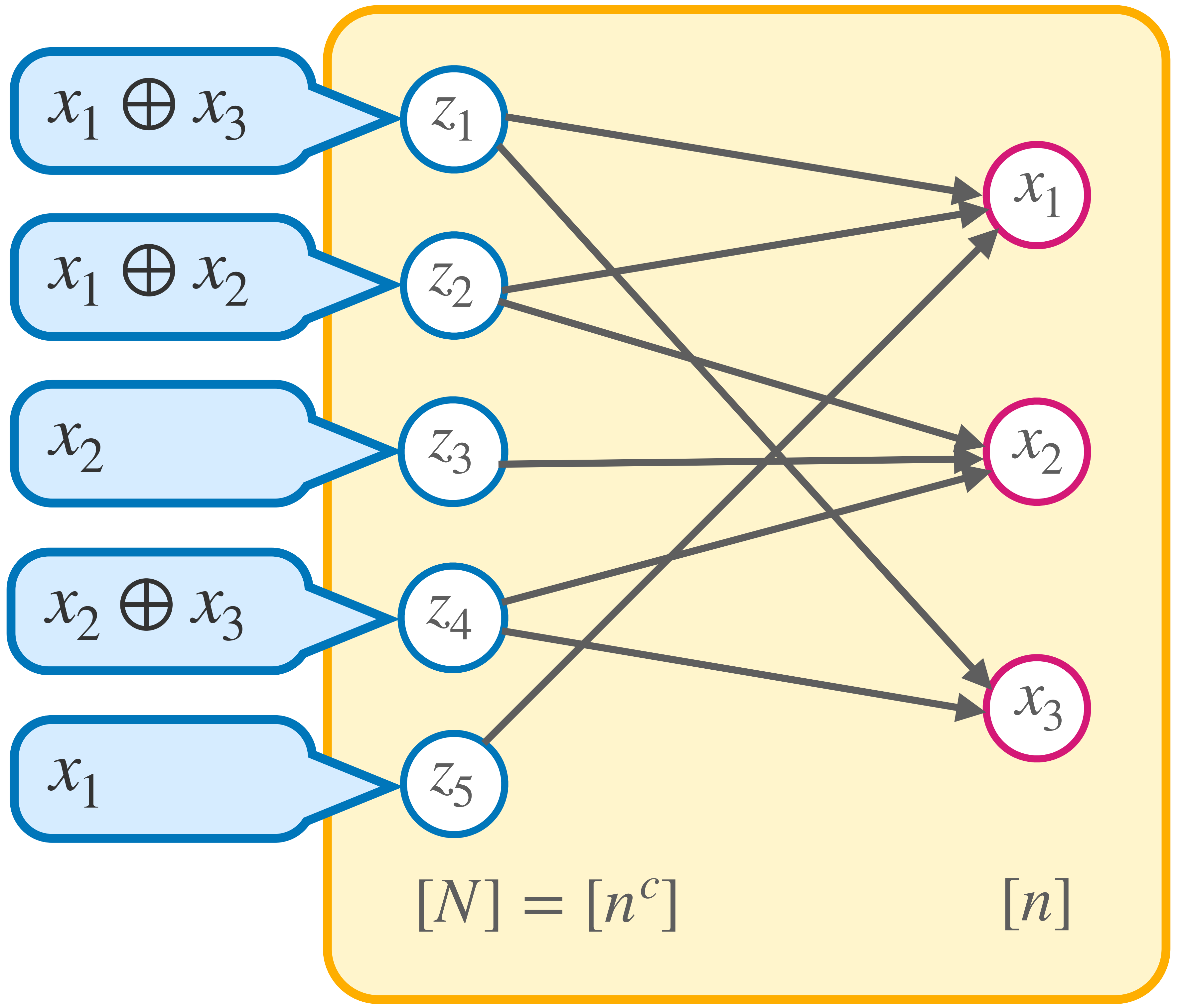

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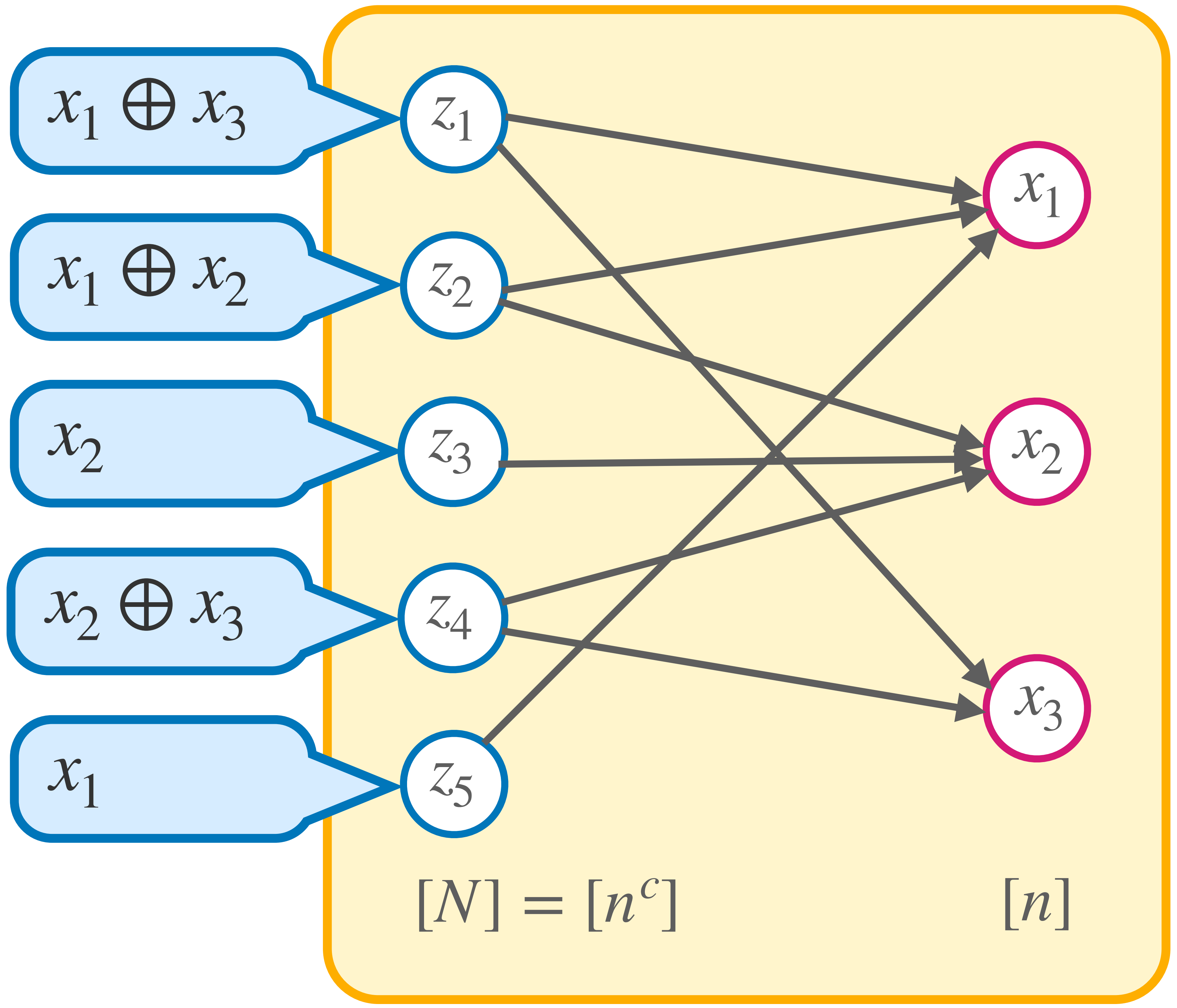

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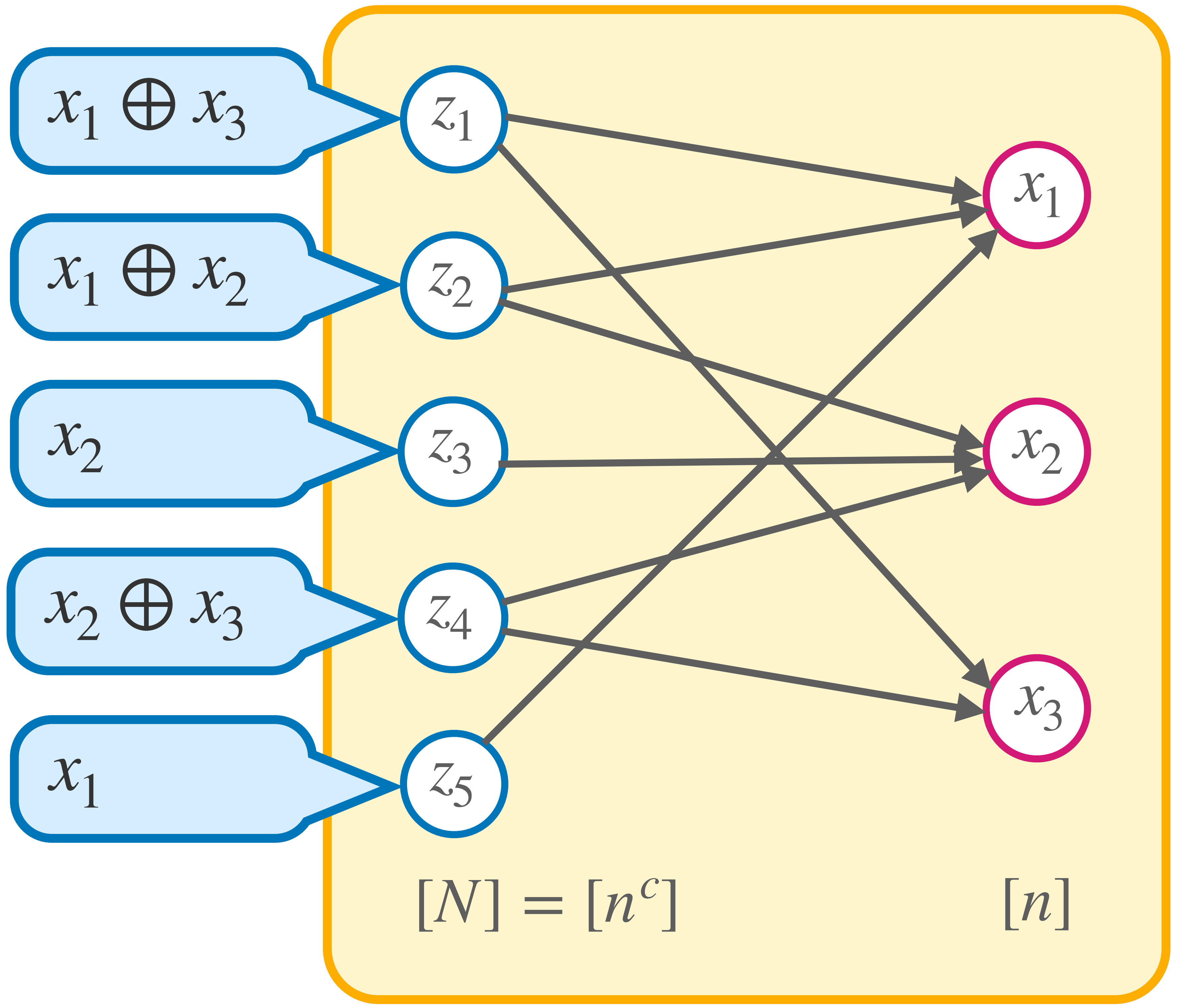

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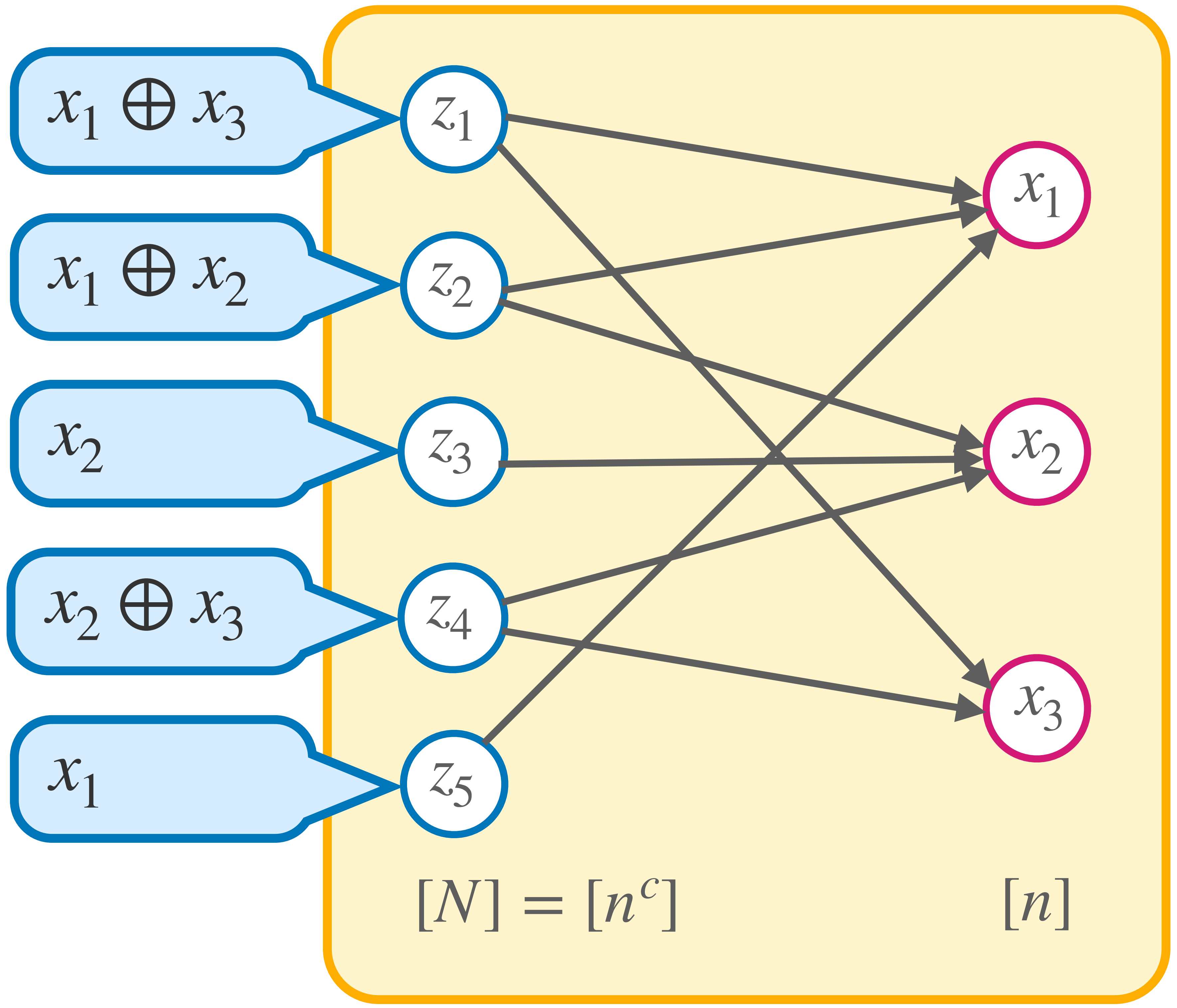

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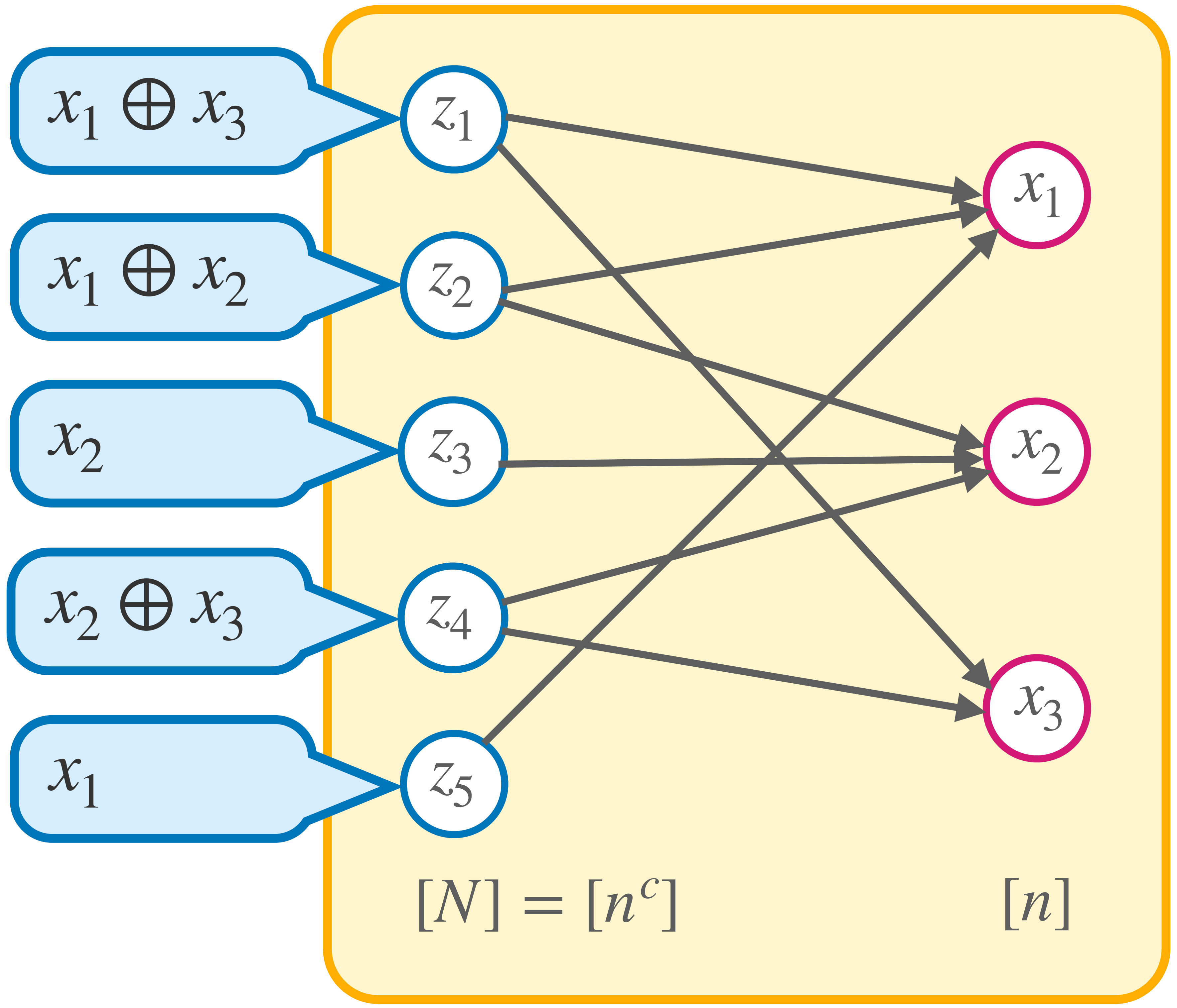

r-Expanding: For any set $U \subseteq [N]$ with $|U| \leq r$ the number of unique neighbours is at least $2|U|$

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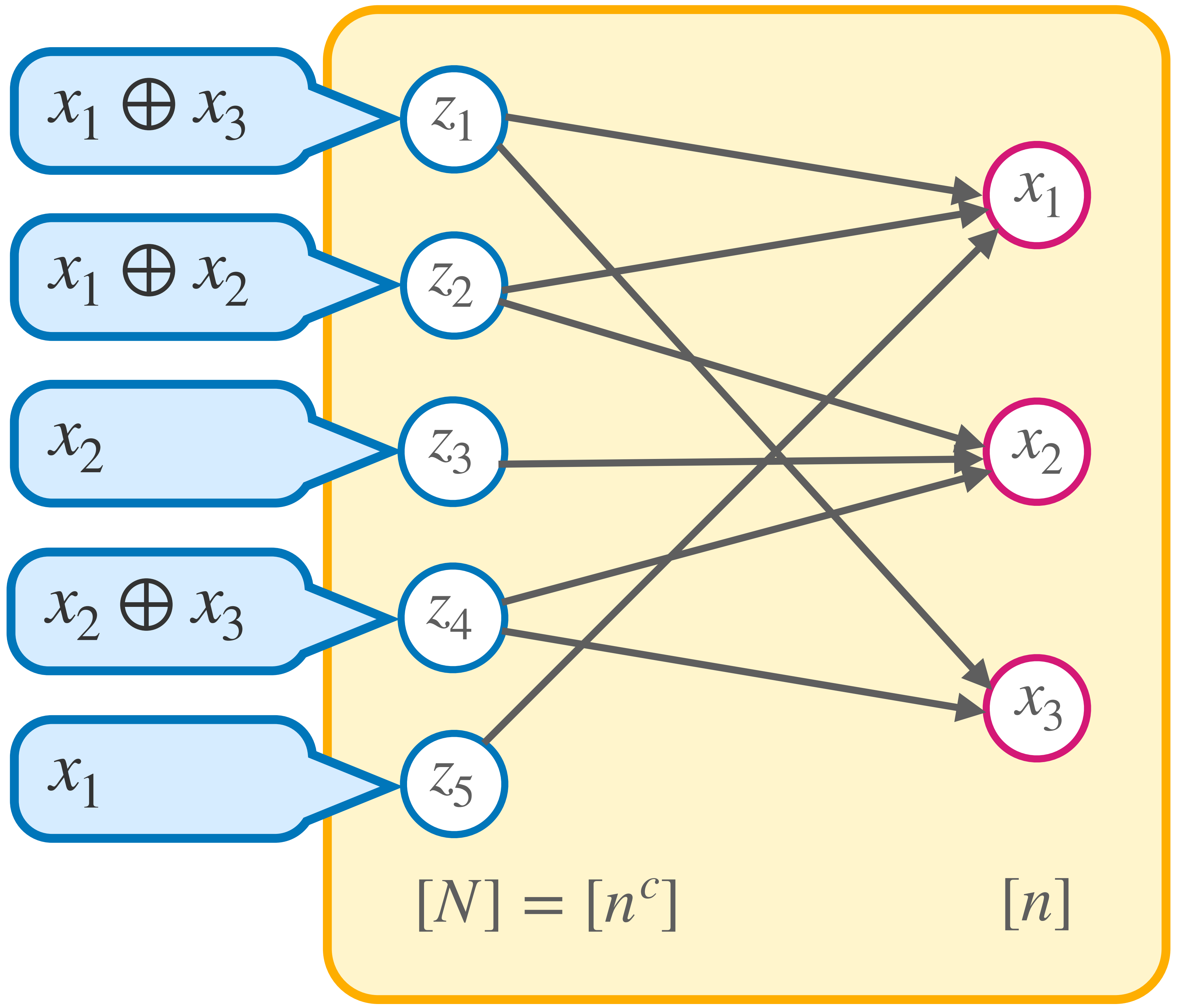

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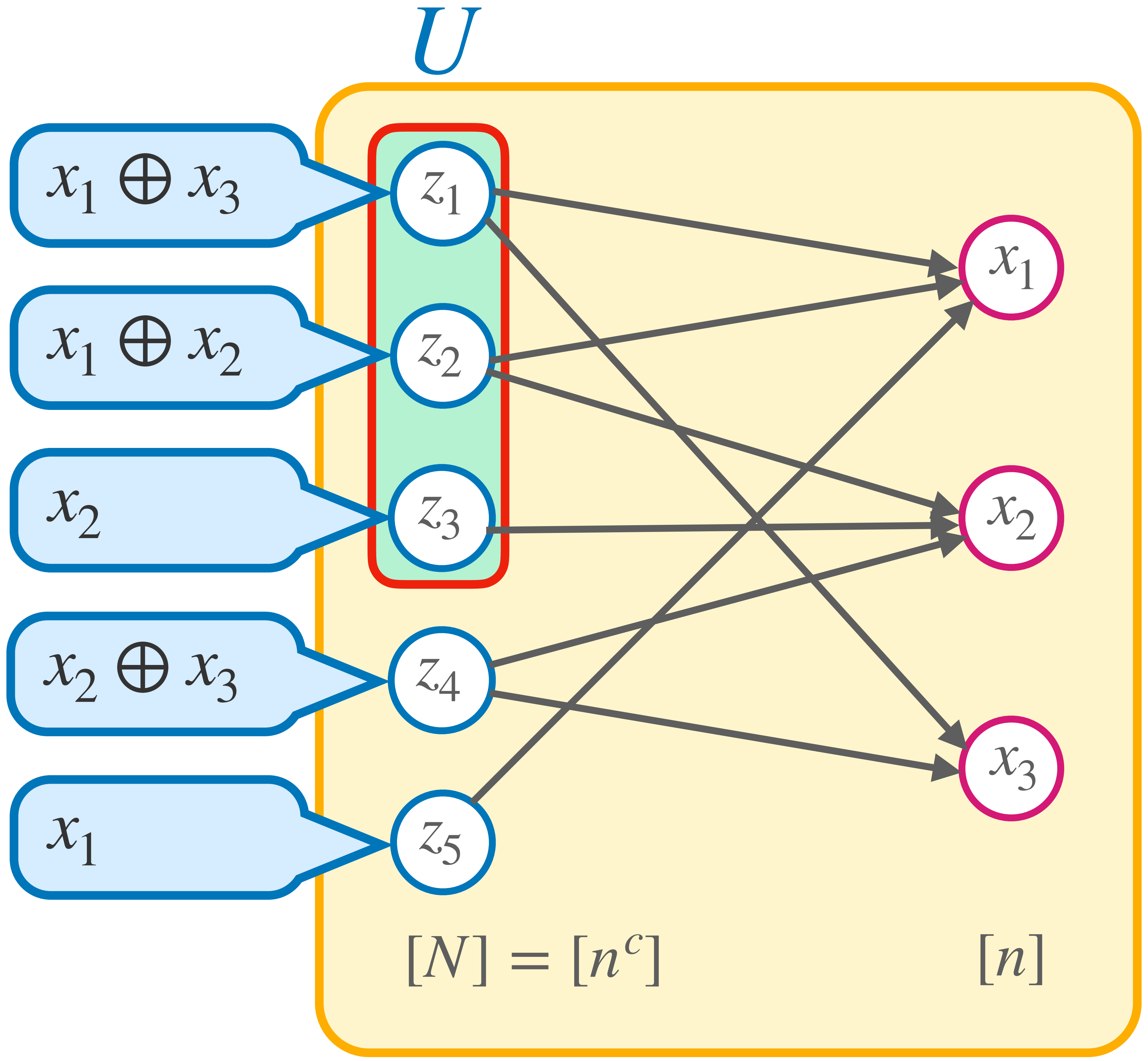

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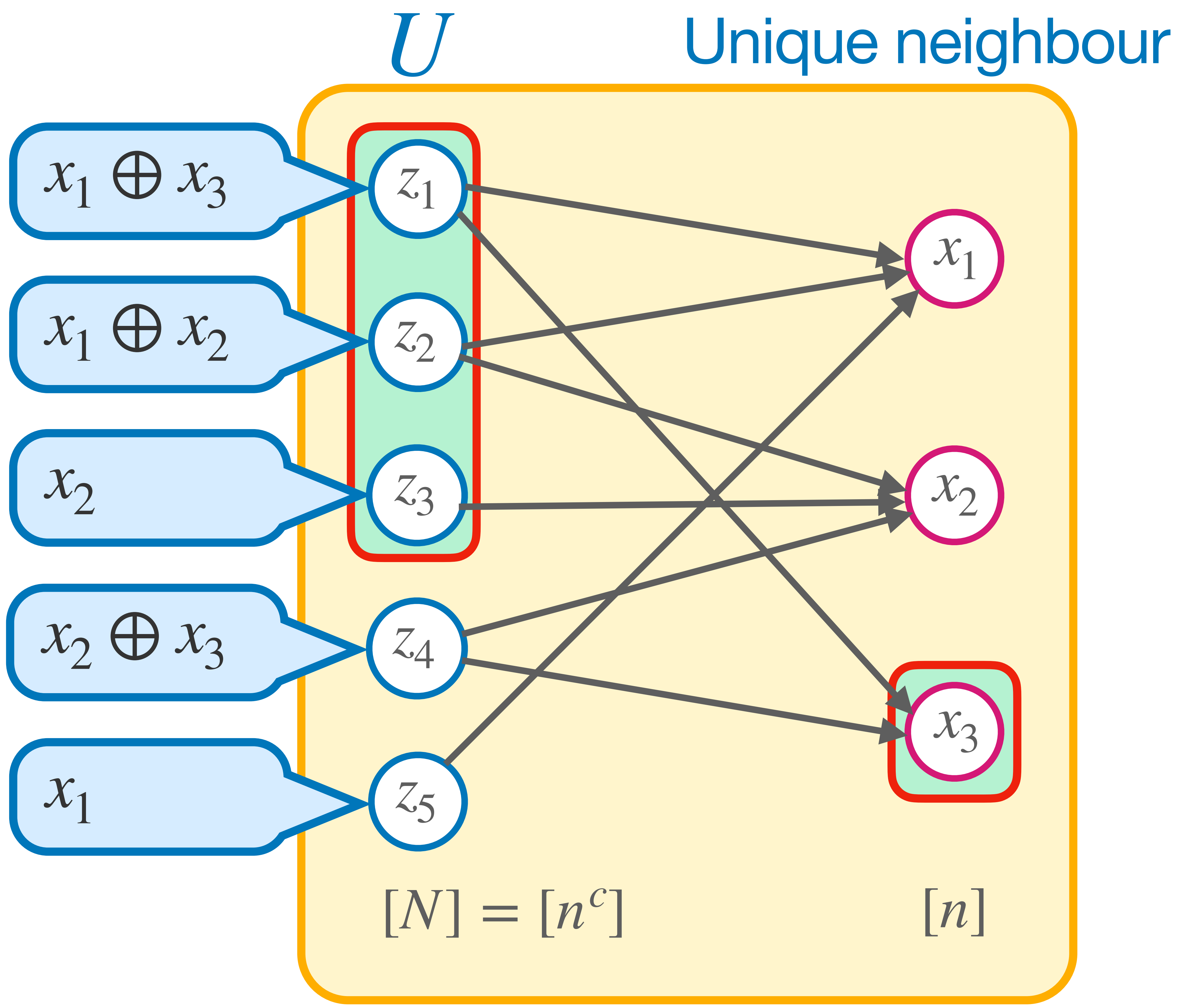

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→ Our gadget g will be XOR_G for expanding G

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Main workhorse behind our tradeoff:

Depth Condensation Theorem: ([Razborov16] stated for tree-resolution)

Let G be r-expanding, F any unsatisfiable formula.

If Π is a Resolution proof of $F \circ \mathsf{XOR}_G$ with $\mathsf{width}(\Pi) \leq r/4$ then

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Let $\varepsilon > 0$, let $c \geq 1$ be real-valued parameter

Main Theorem: There is a CNF formula F on n variables such that

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- For Res(k) we prove a Resolution width $\rightarrow$ Res(k) size lifting theorem with $g = XOR_2$, which uses the switching lemma of **[SBI04]**

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Our proof uses a characterization of resolution depth by Prover-Adversary games

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Claim: If there is a strategy for the Adversary such that the game always continues for at least d rounds, then any resolution proof of F requires depth $\geq d$

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Unbounded Game: No bound on $|\rho|$

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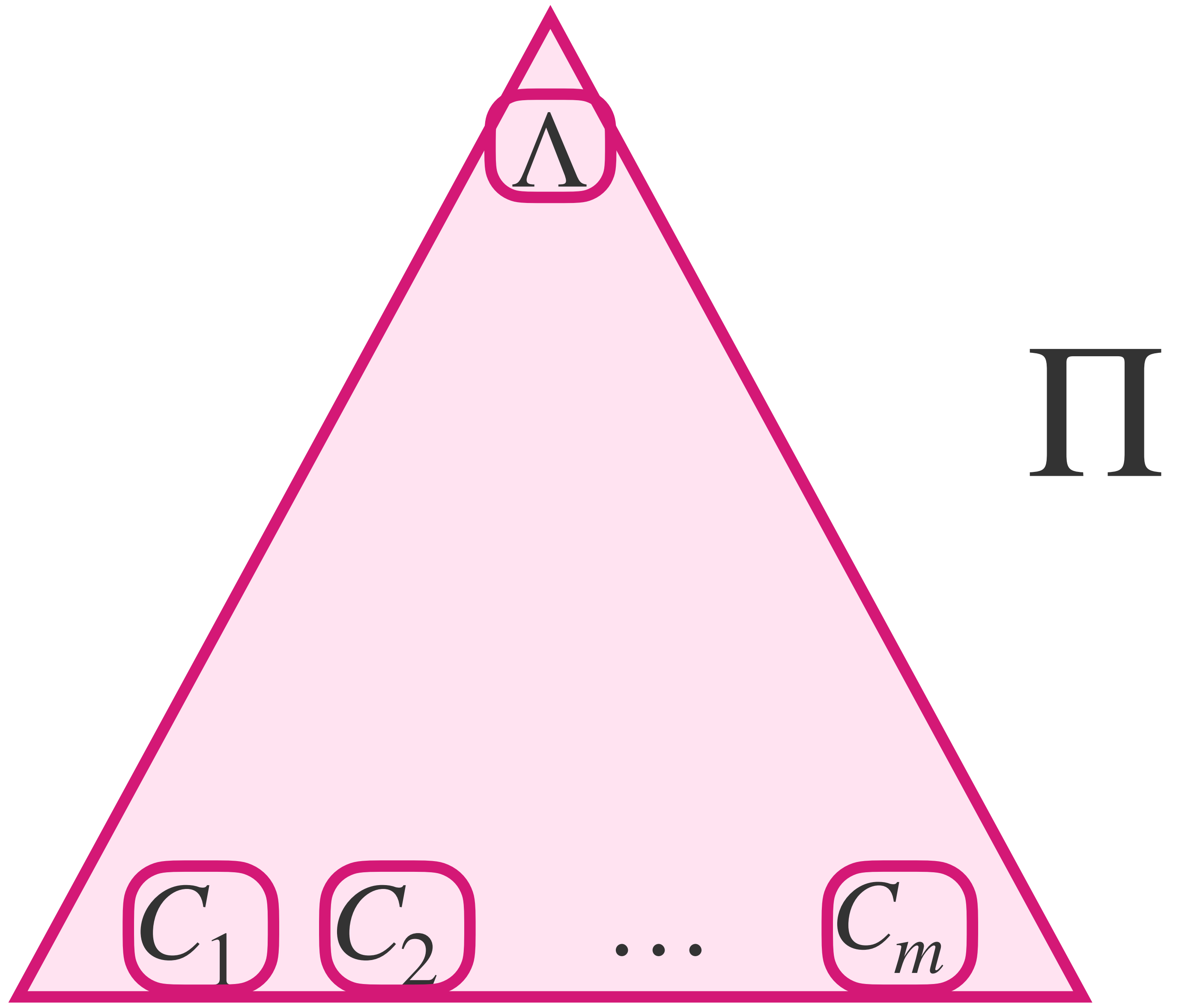

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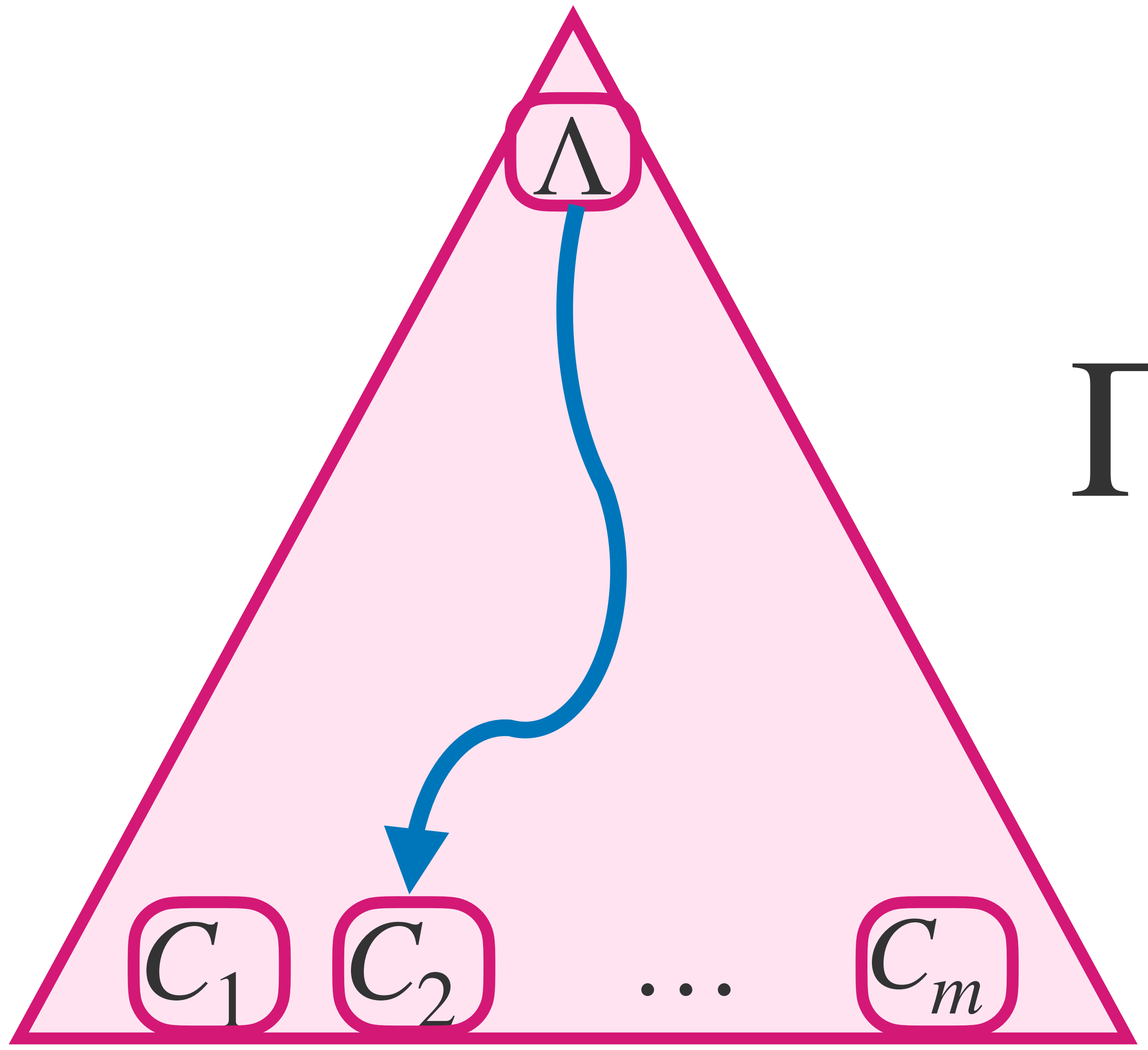

Π

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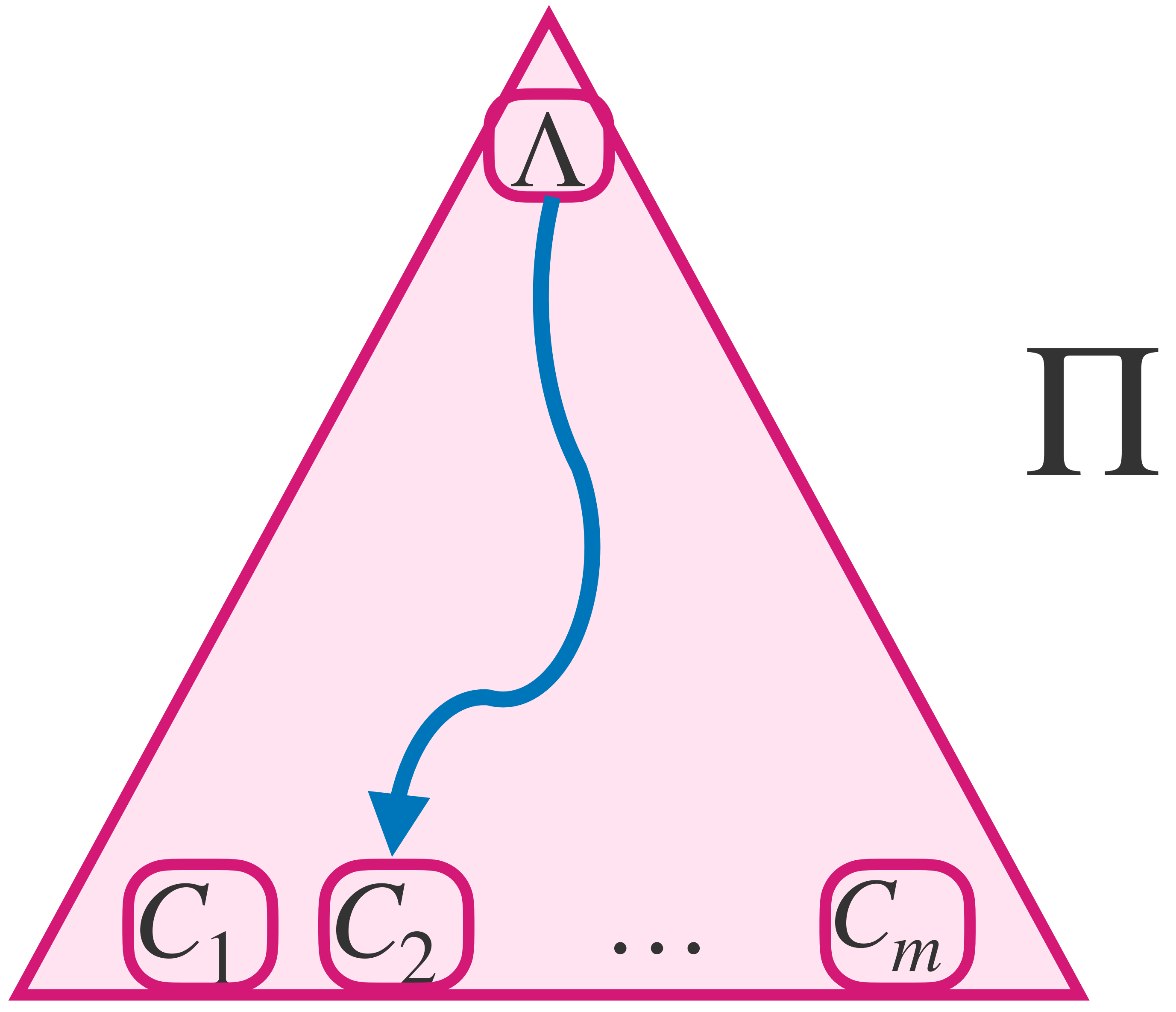

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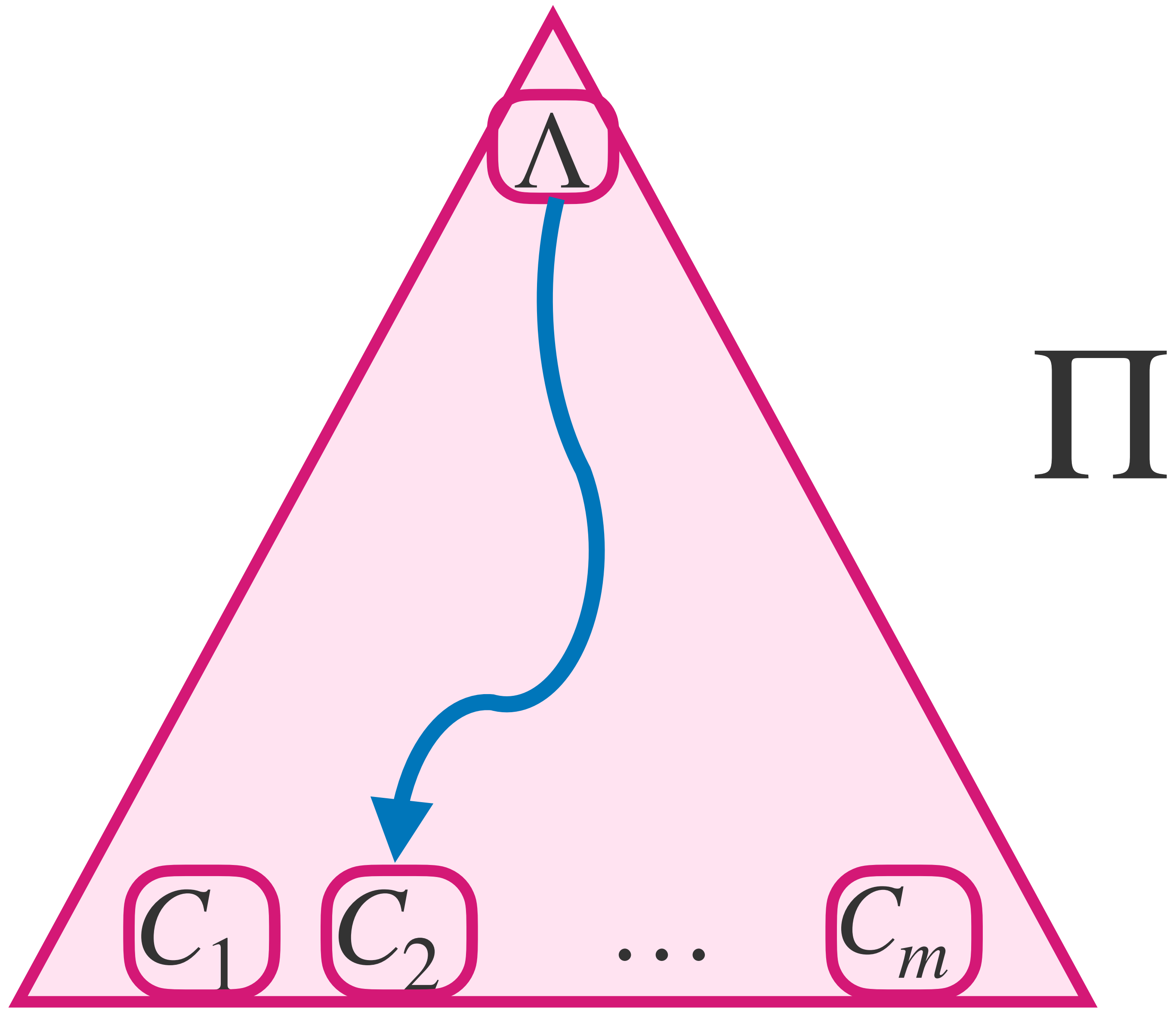

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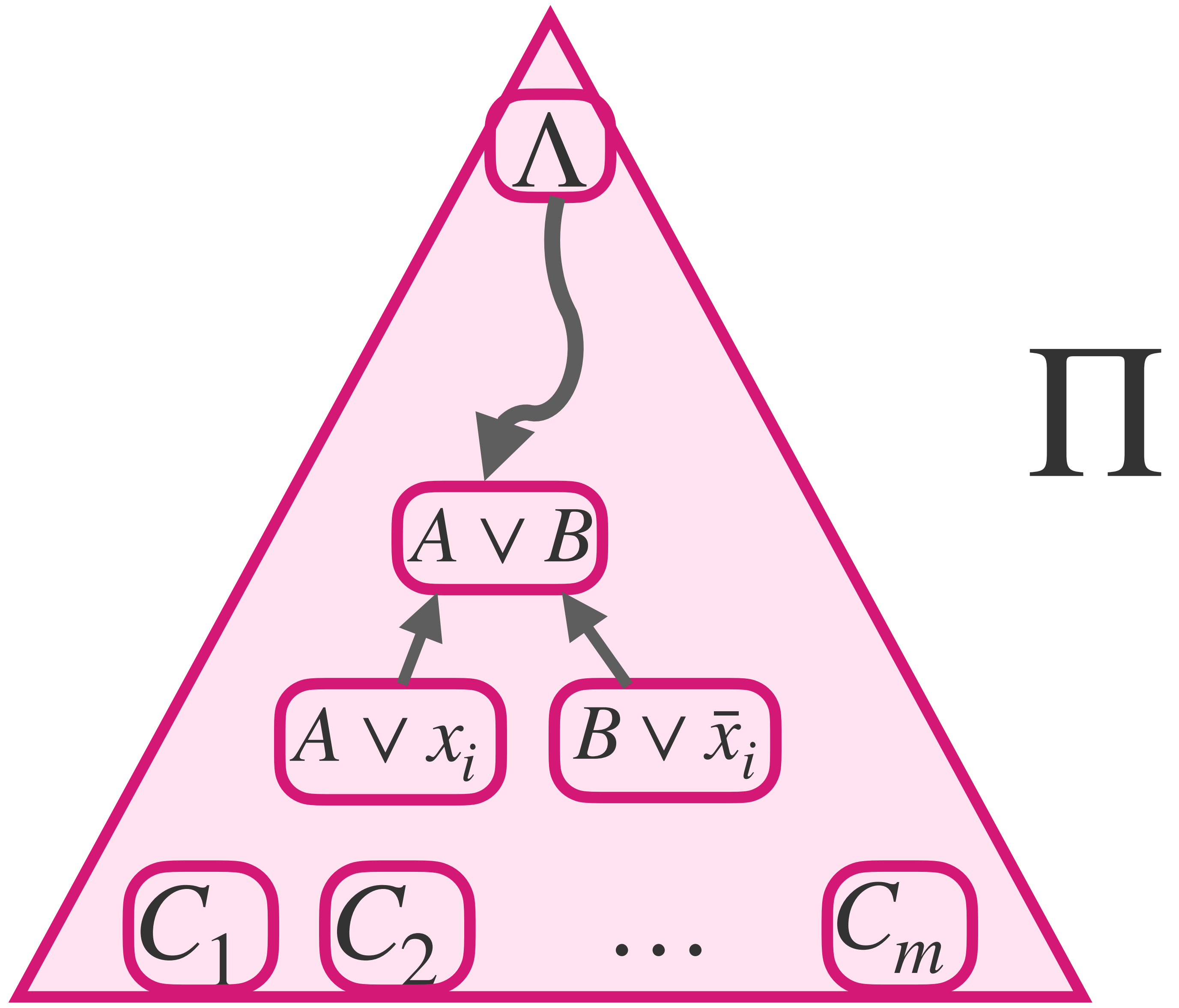

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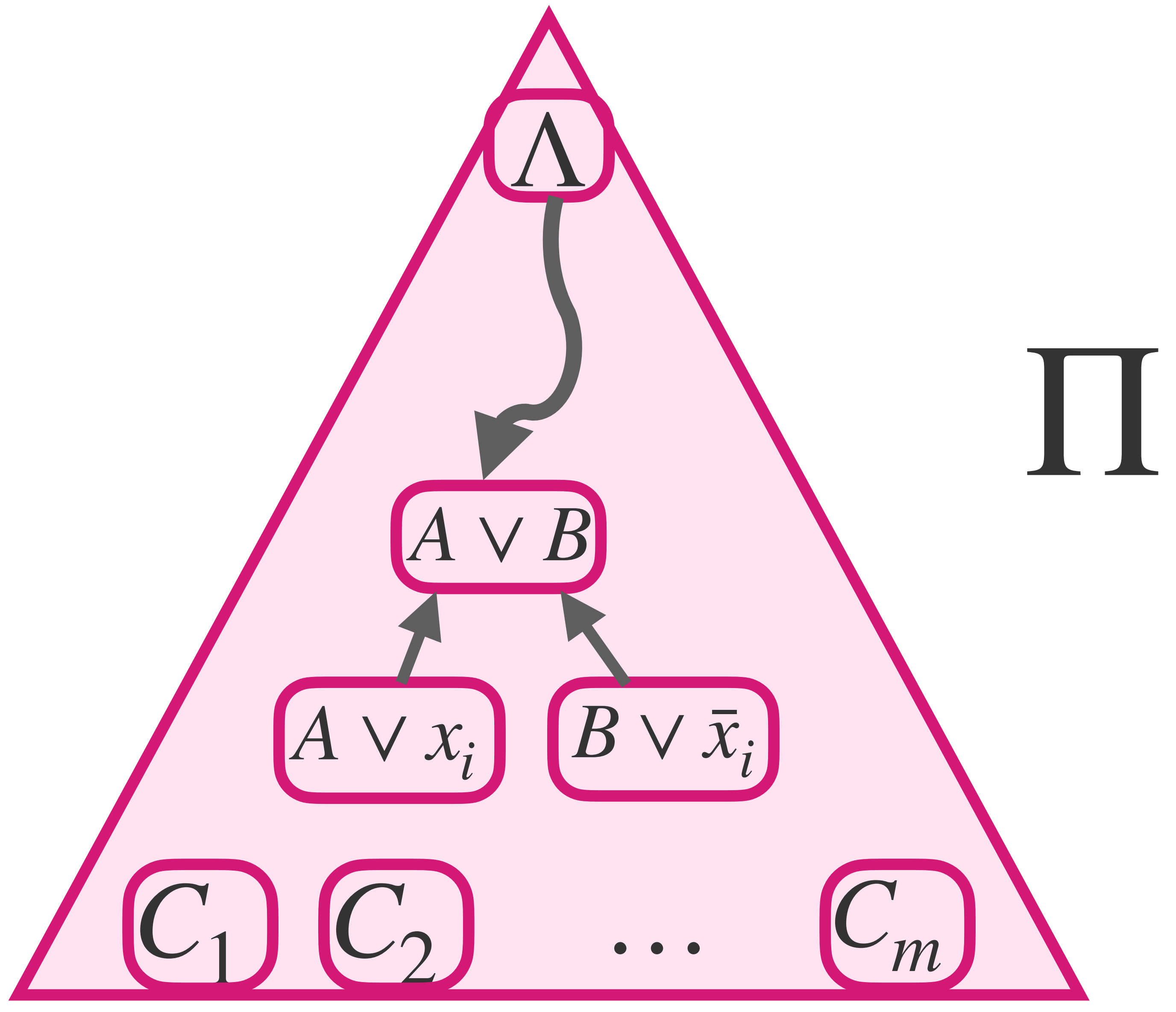

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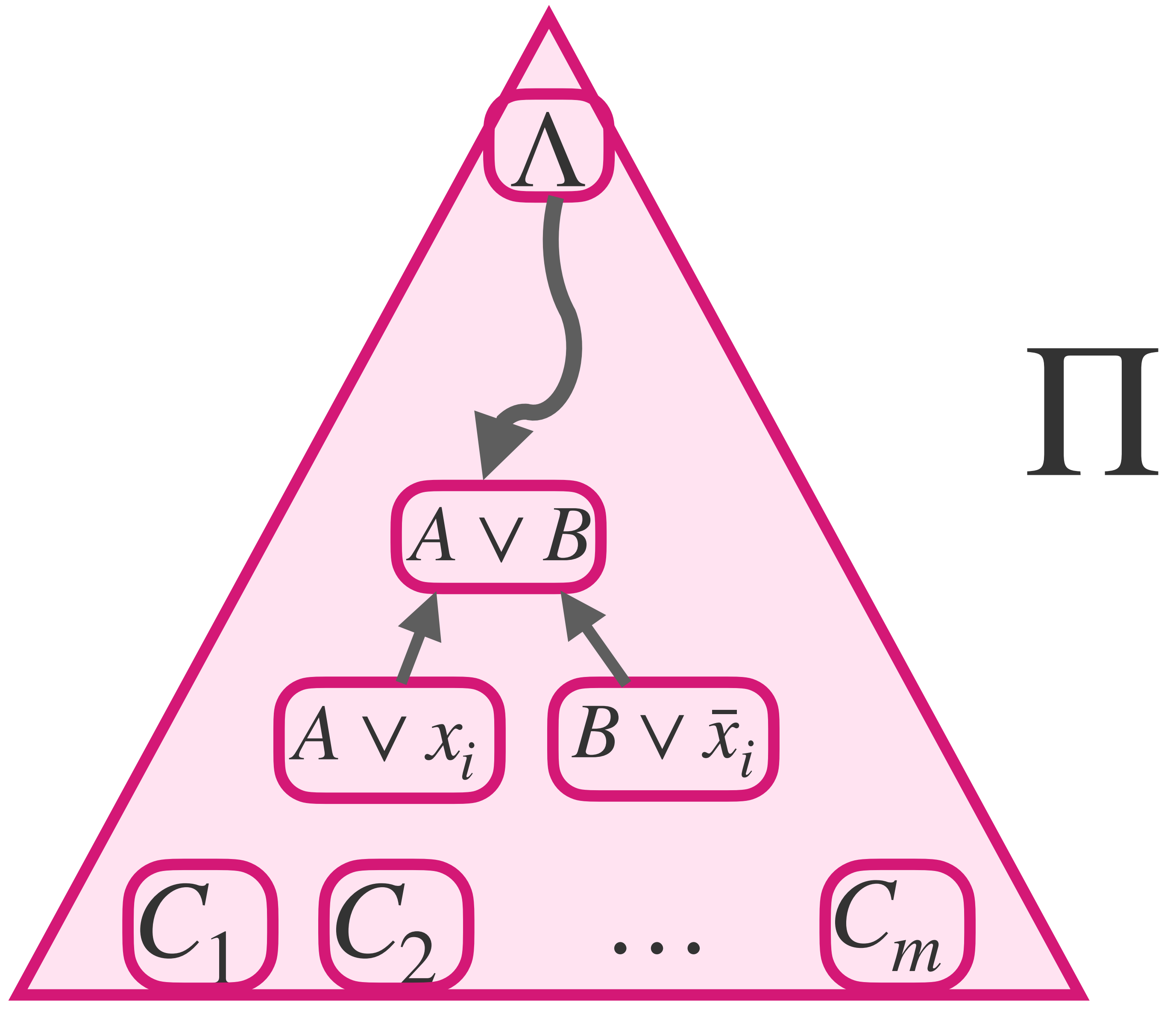

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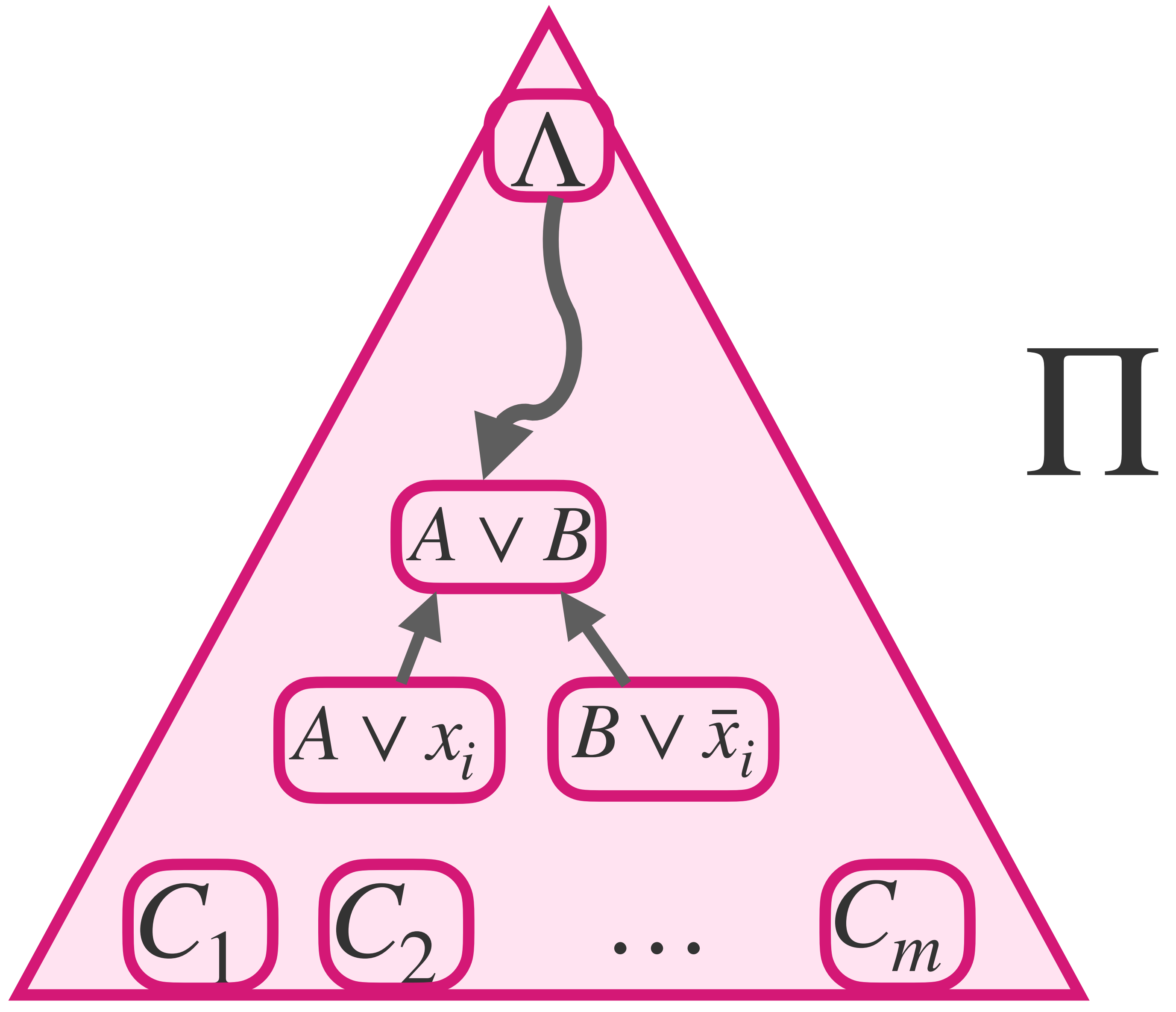
- Otherwise, move to $B \vee \bar{x}_i$. Forget $A \backslash B$



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Let G be an r-boundary expander, F any unsatisfiable formula.

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$\rightarrow$ Use A to construct an Adversary Strategy for the w-bounded game on $F \circ XOR_G$ to survive $\Omega(d/w)$ rounds, for any $w \leq r/4$.

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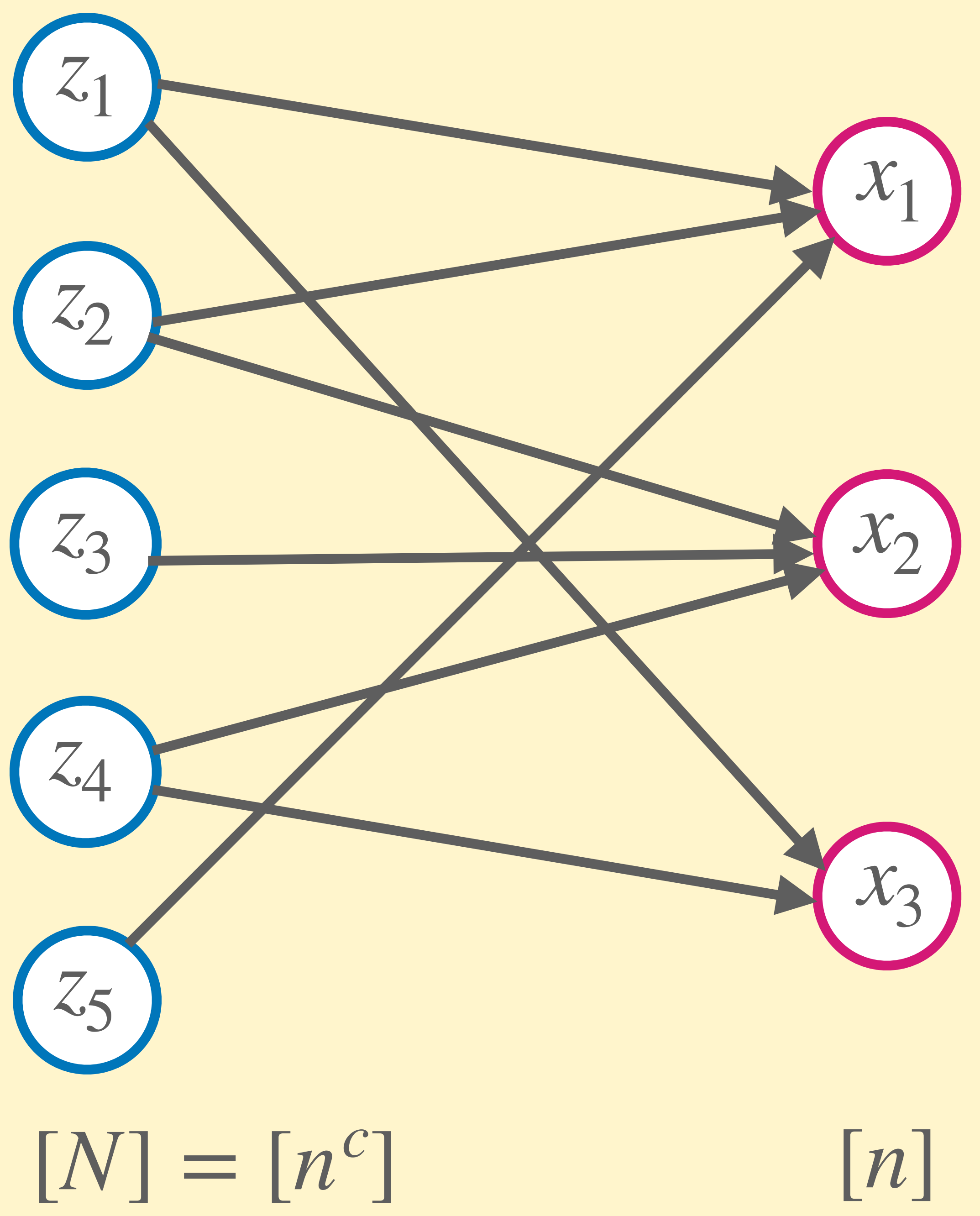

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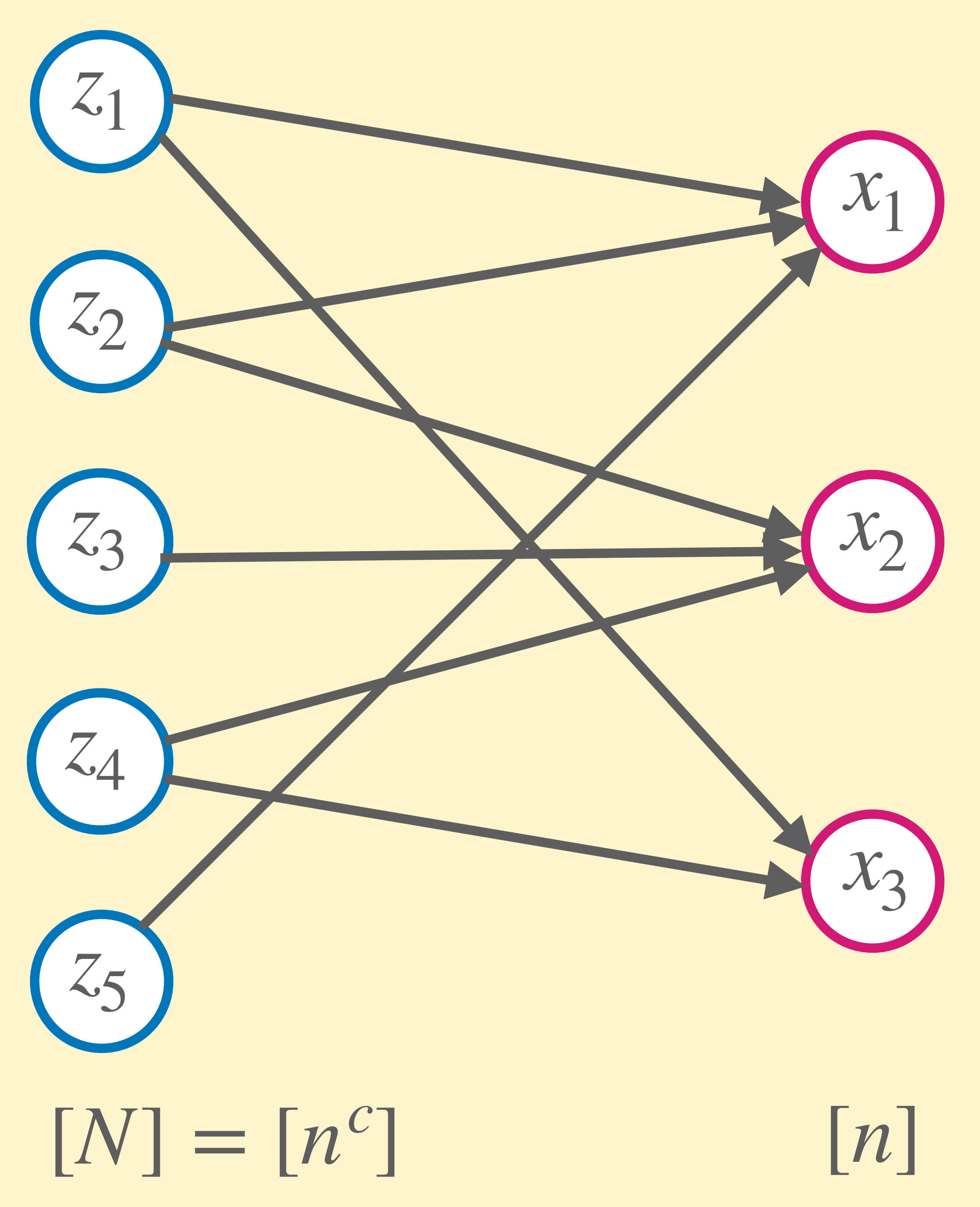

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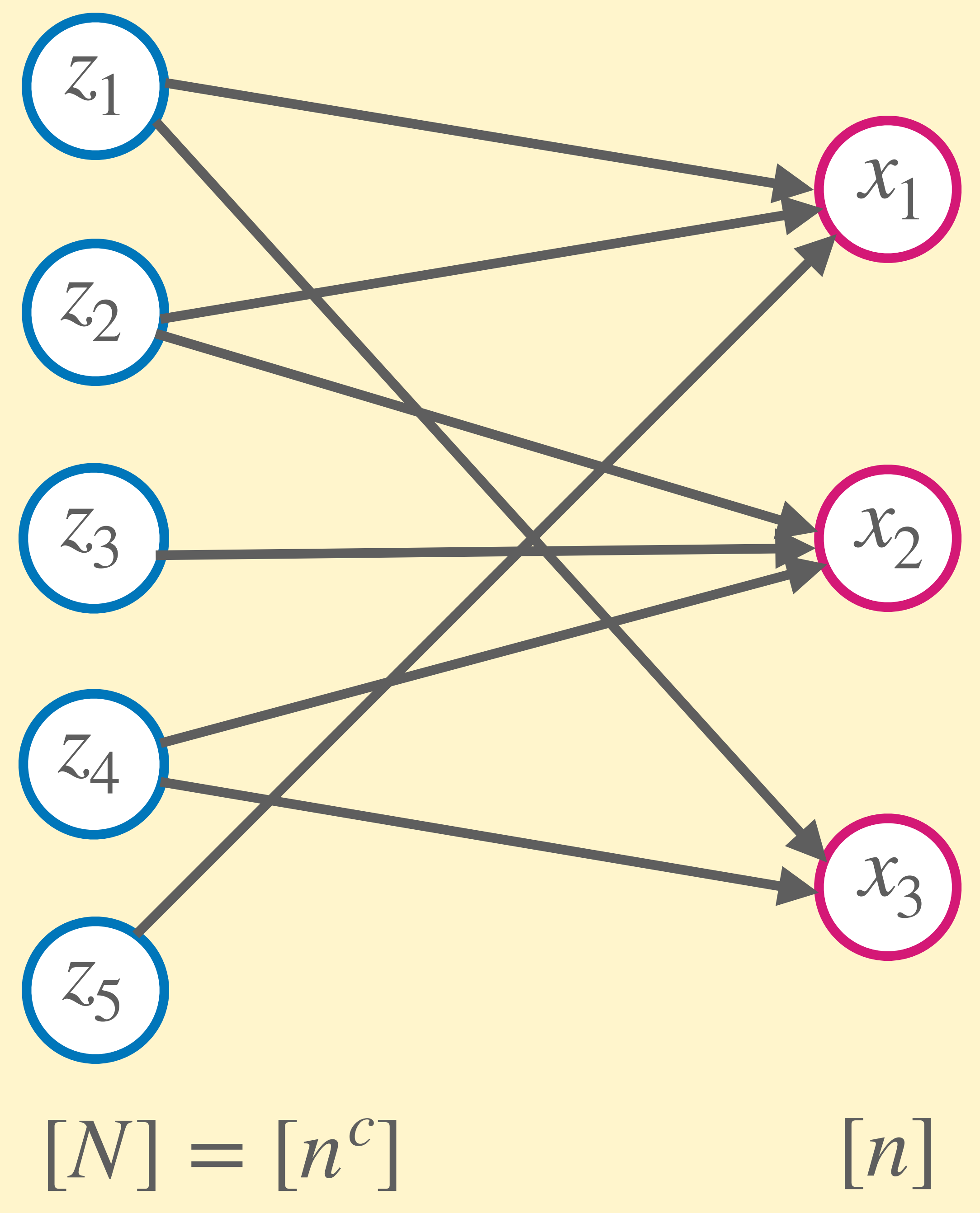

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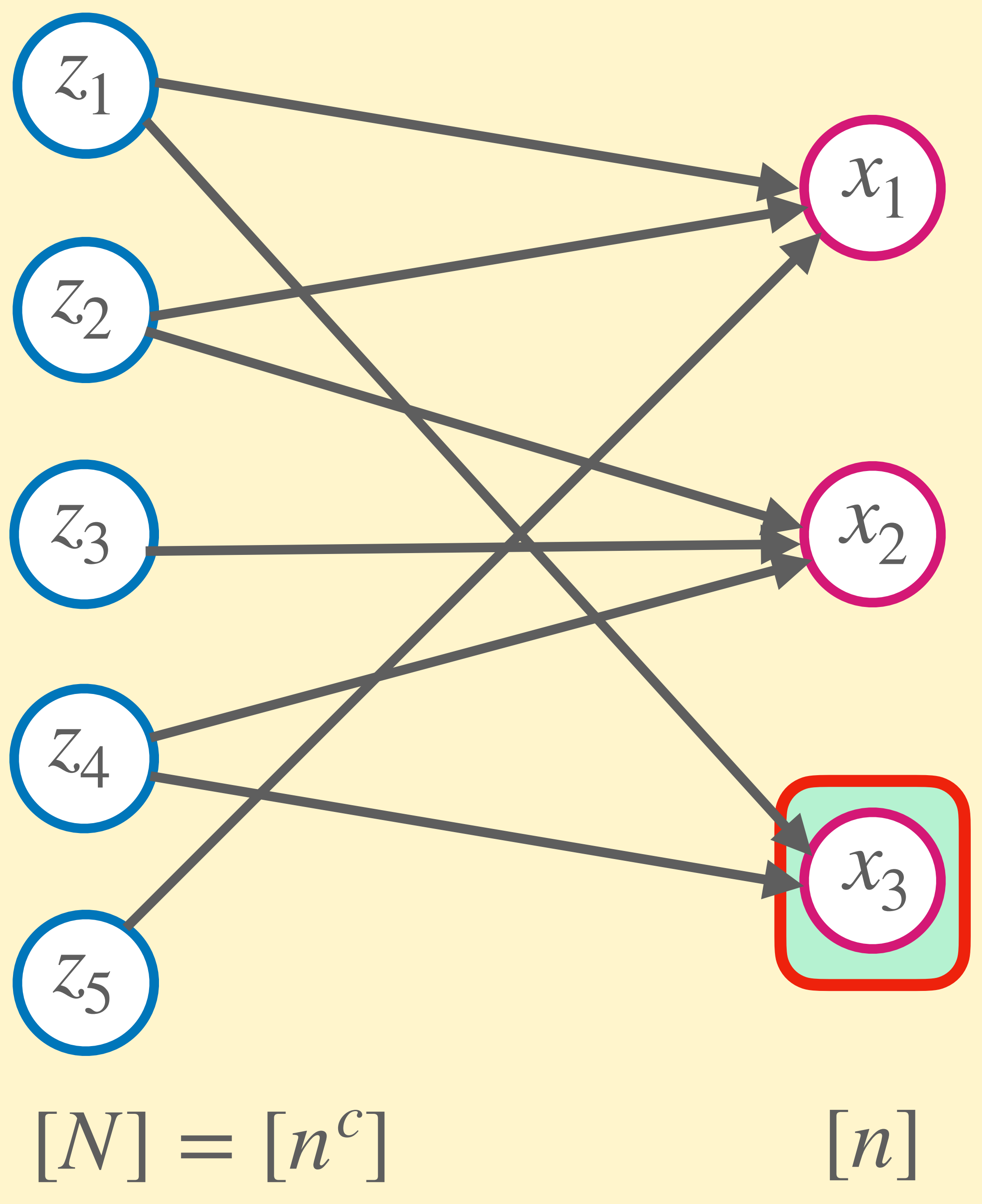

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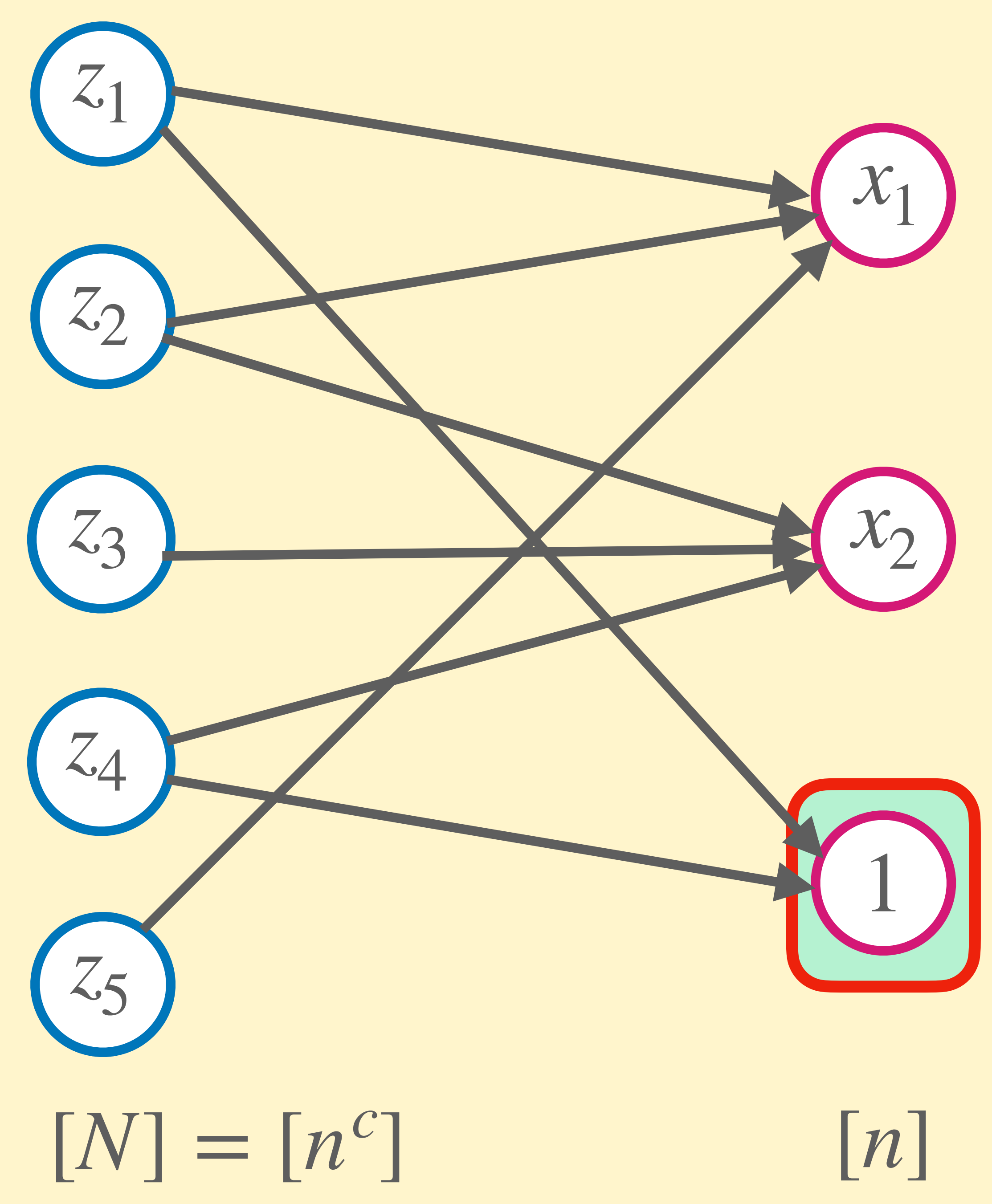

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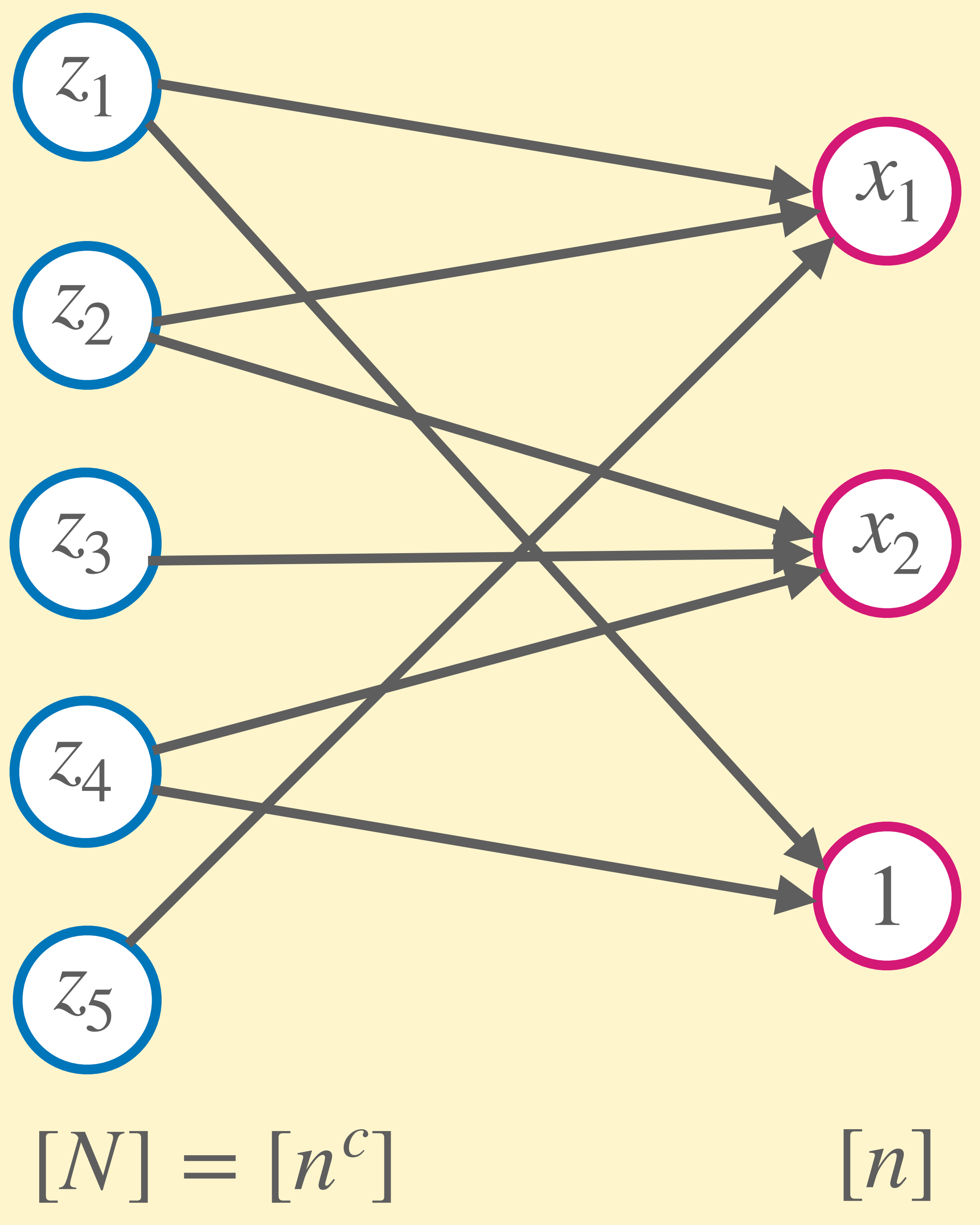

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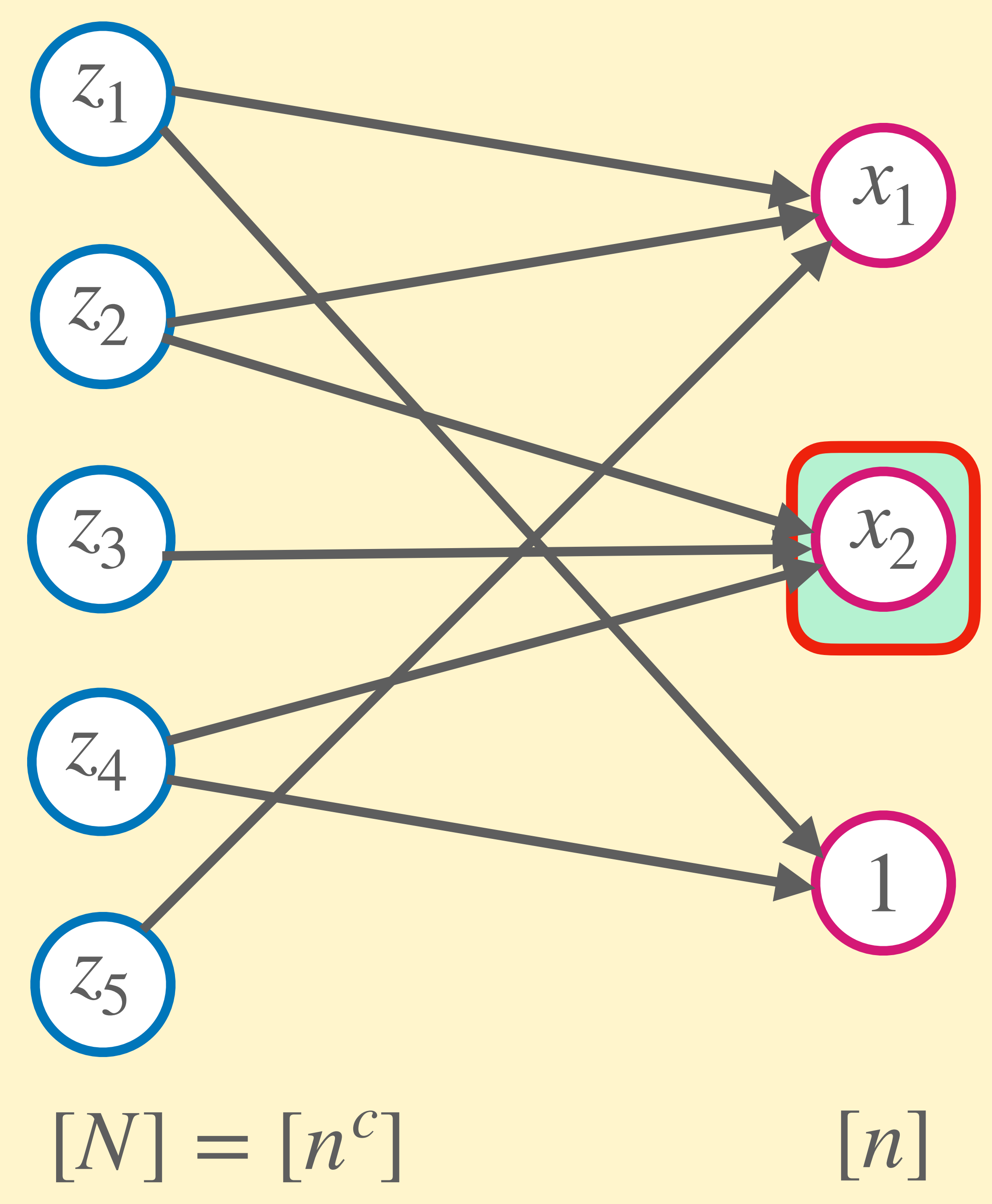

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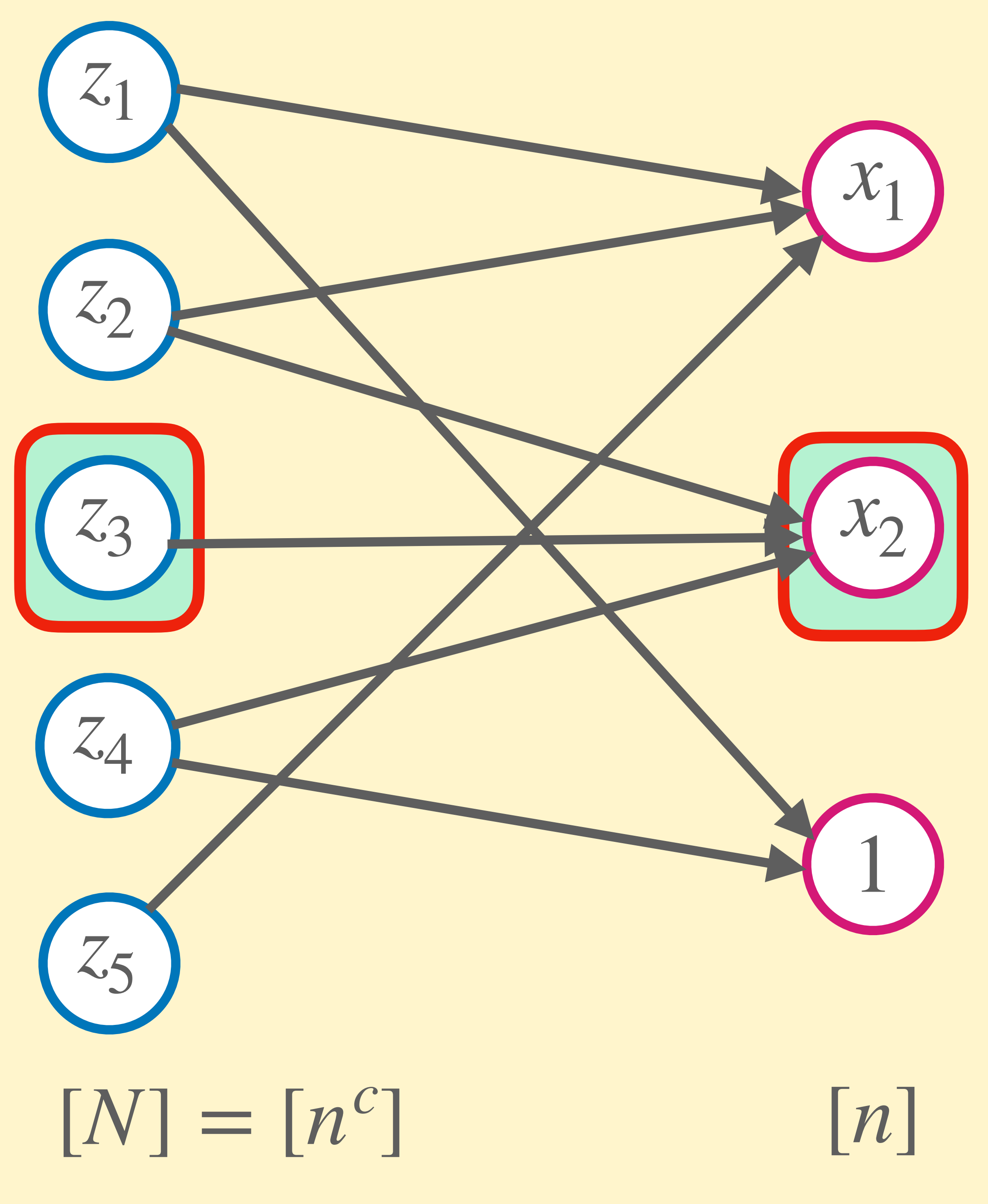

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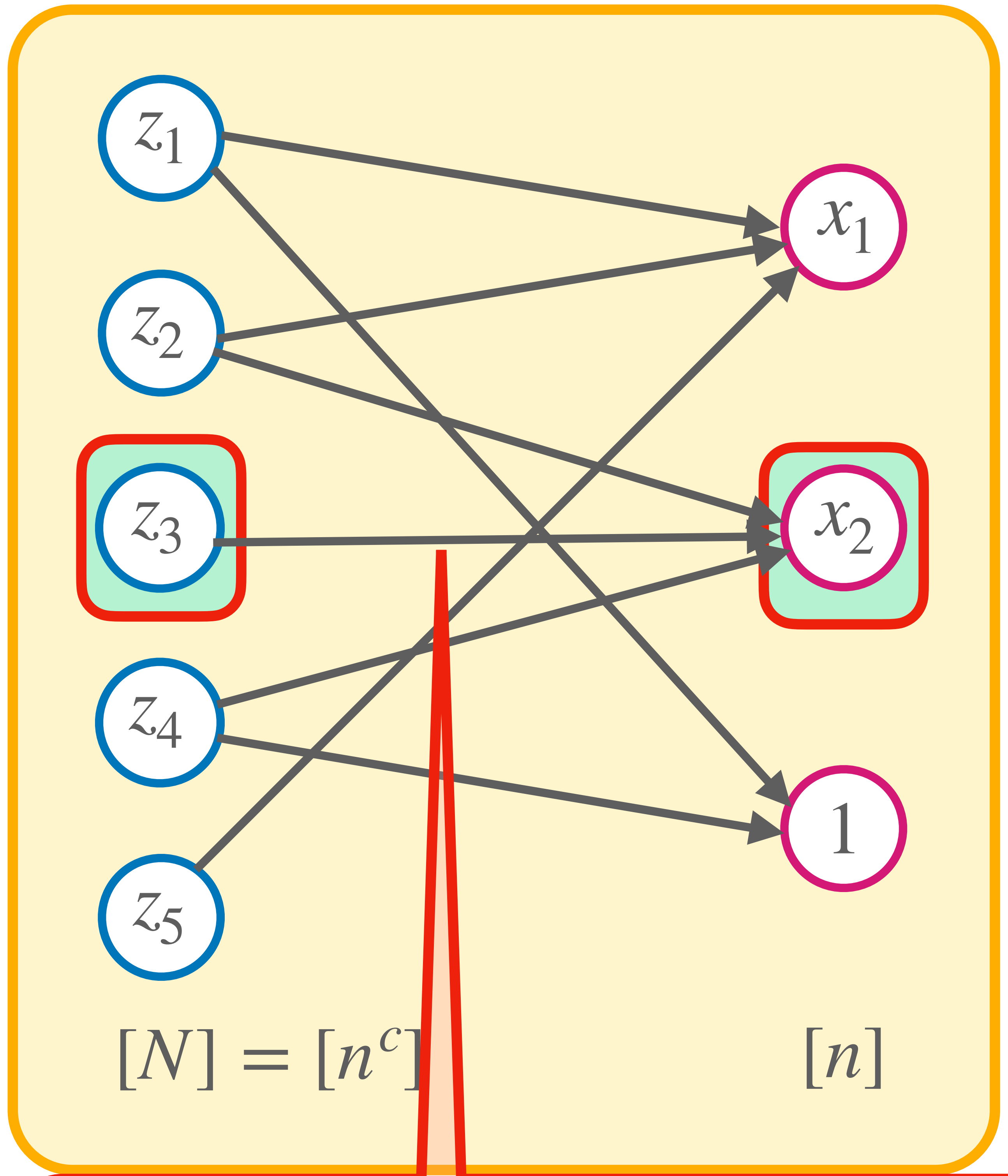

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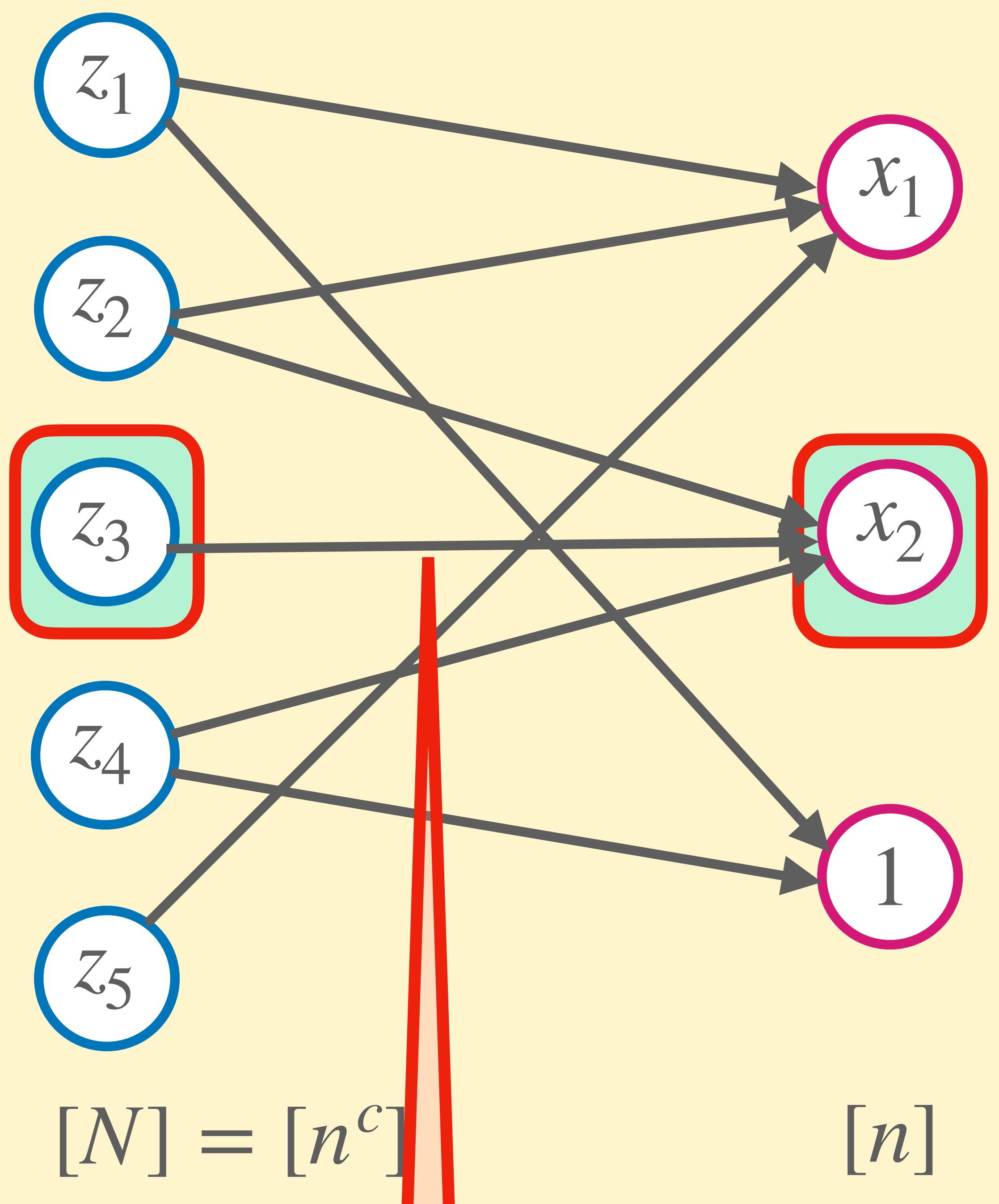

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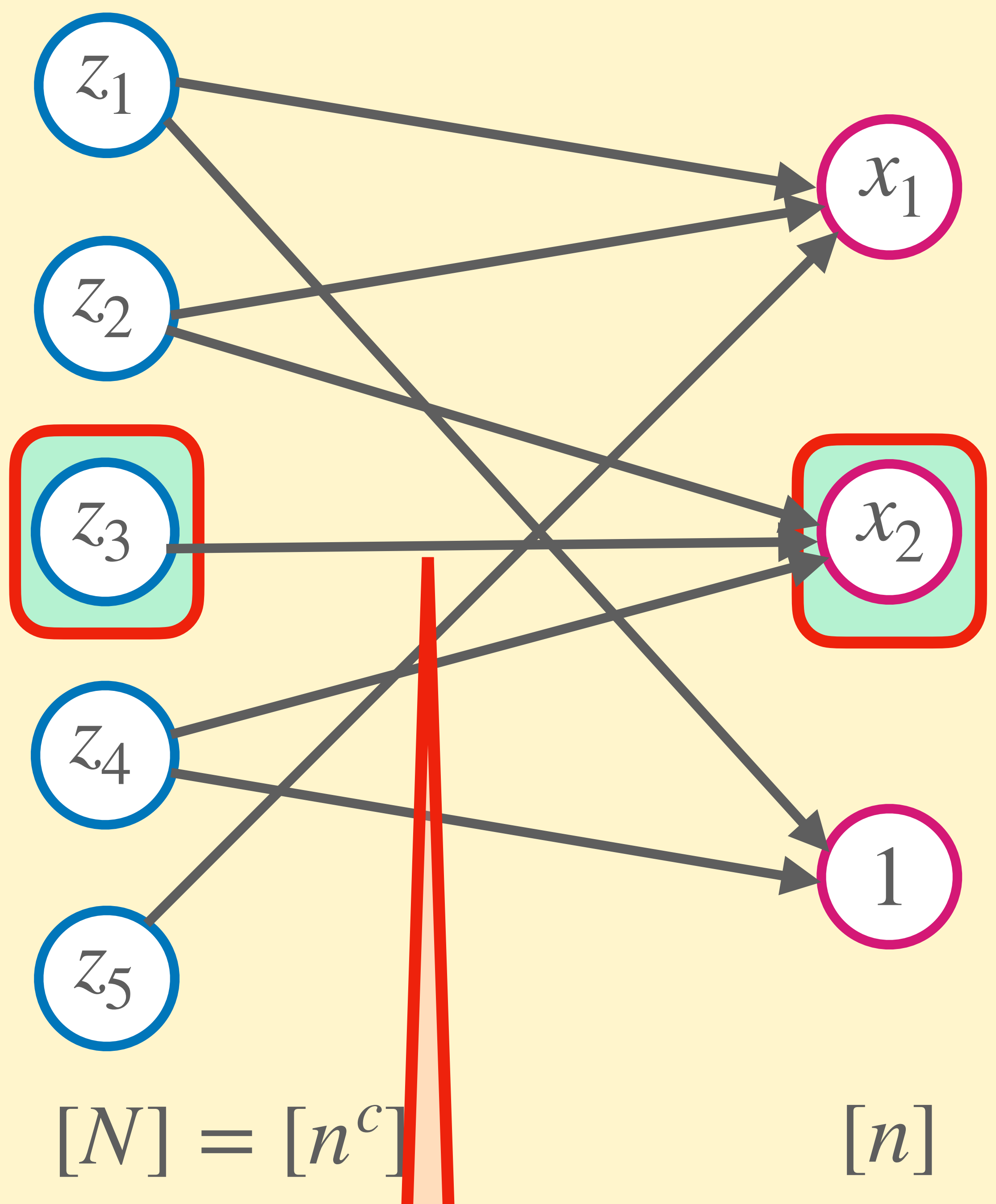

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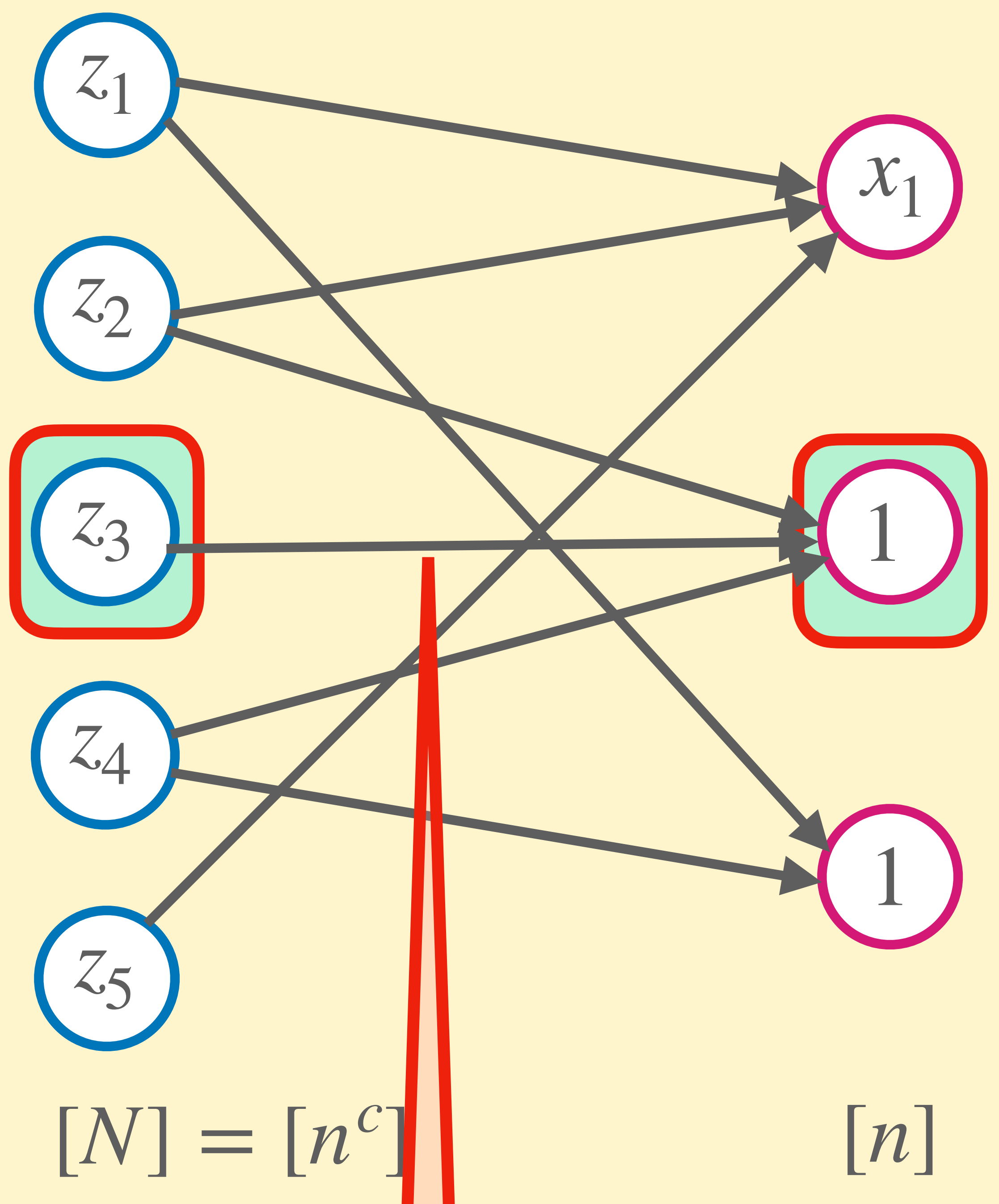

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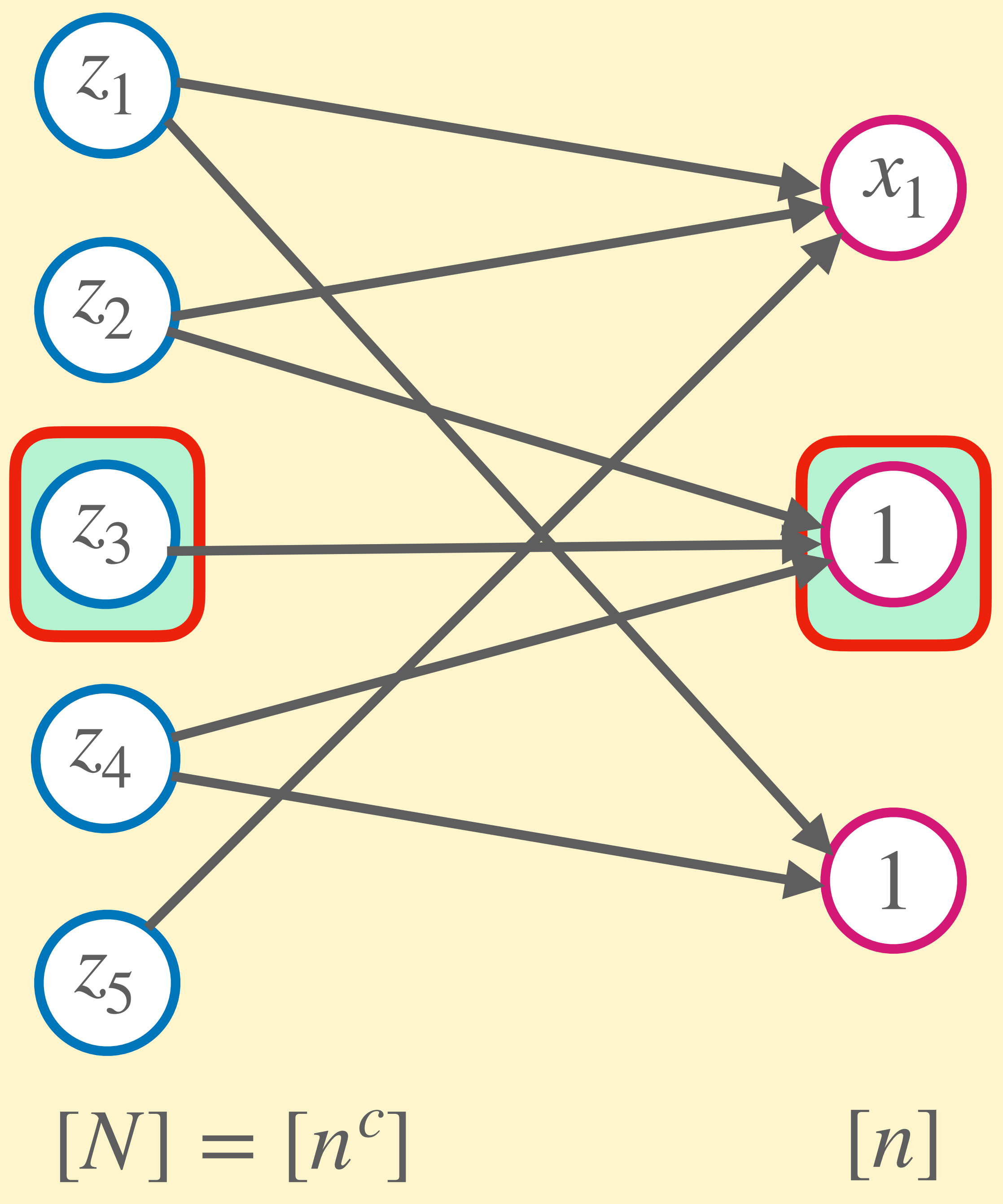


Problem!

This forces $z_4 = 0$

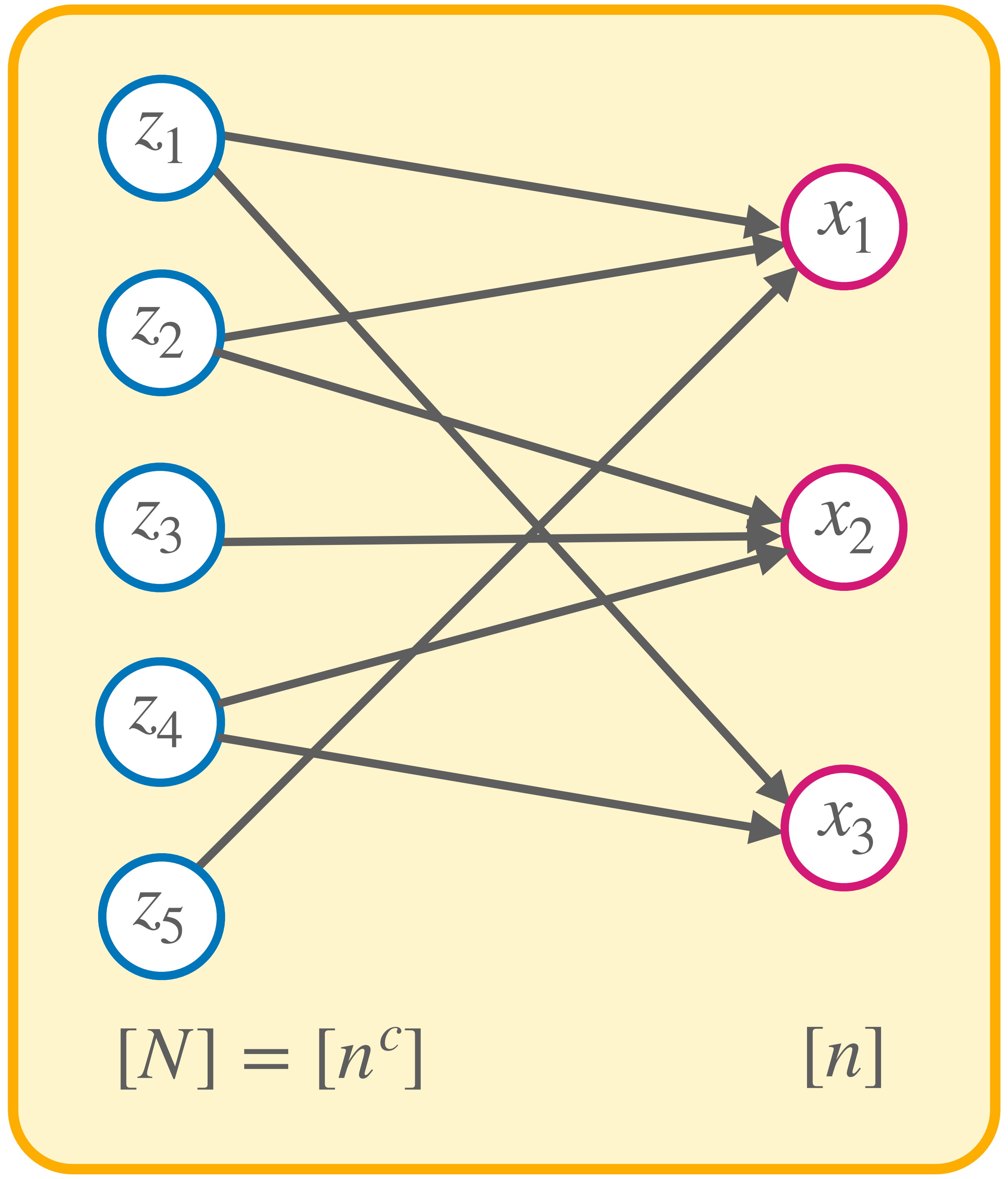
What if A sets $z_4 = 1$?

Proof Overview

Problem: z-variables are correlated

$\rightarrow$ Setting one can x-variable can force several z-variables

$\rightarrow$ Cannot follow A in this case



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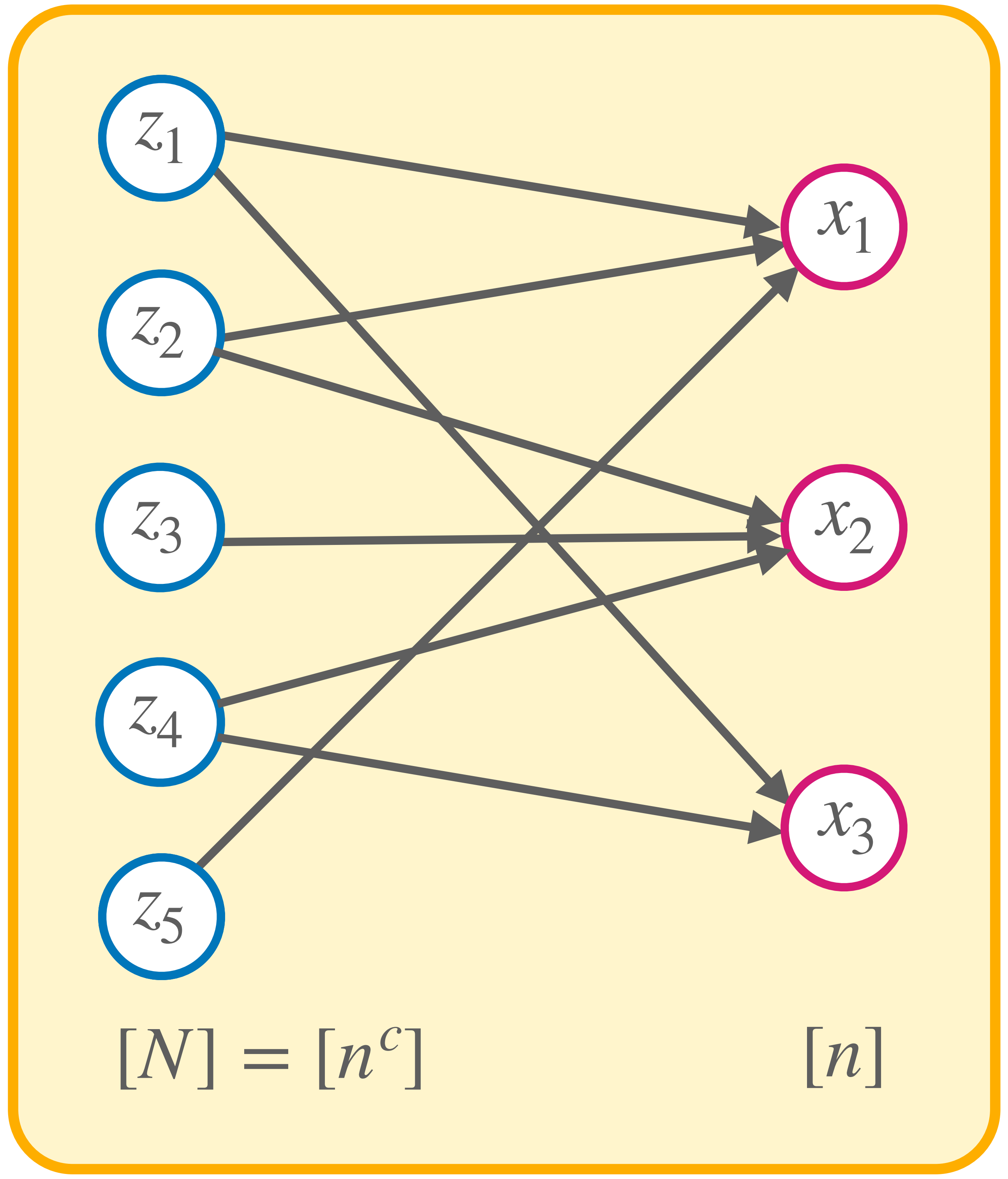

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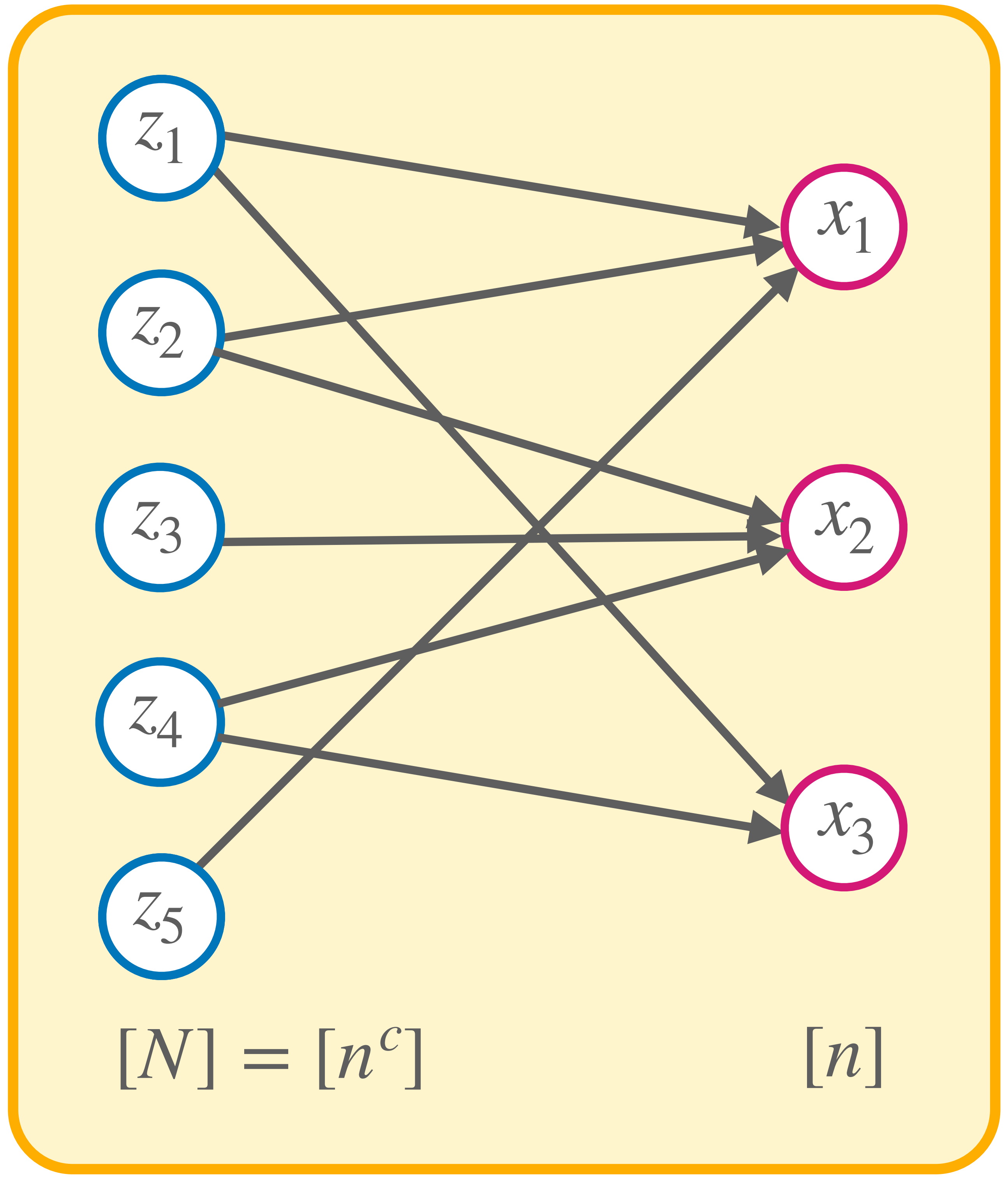

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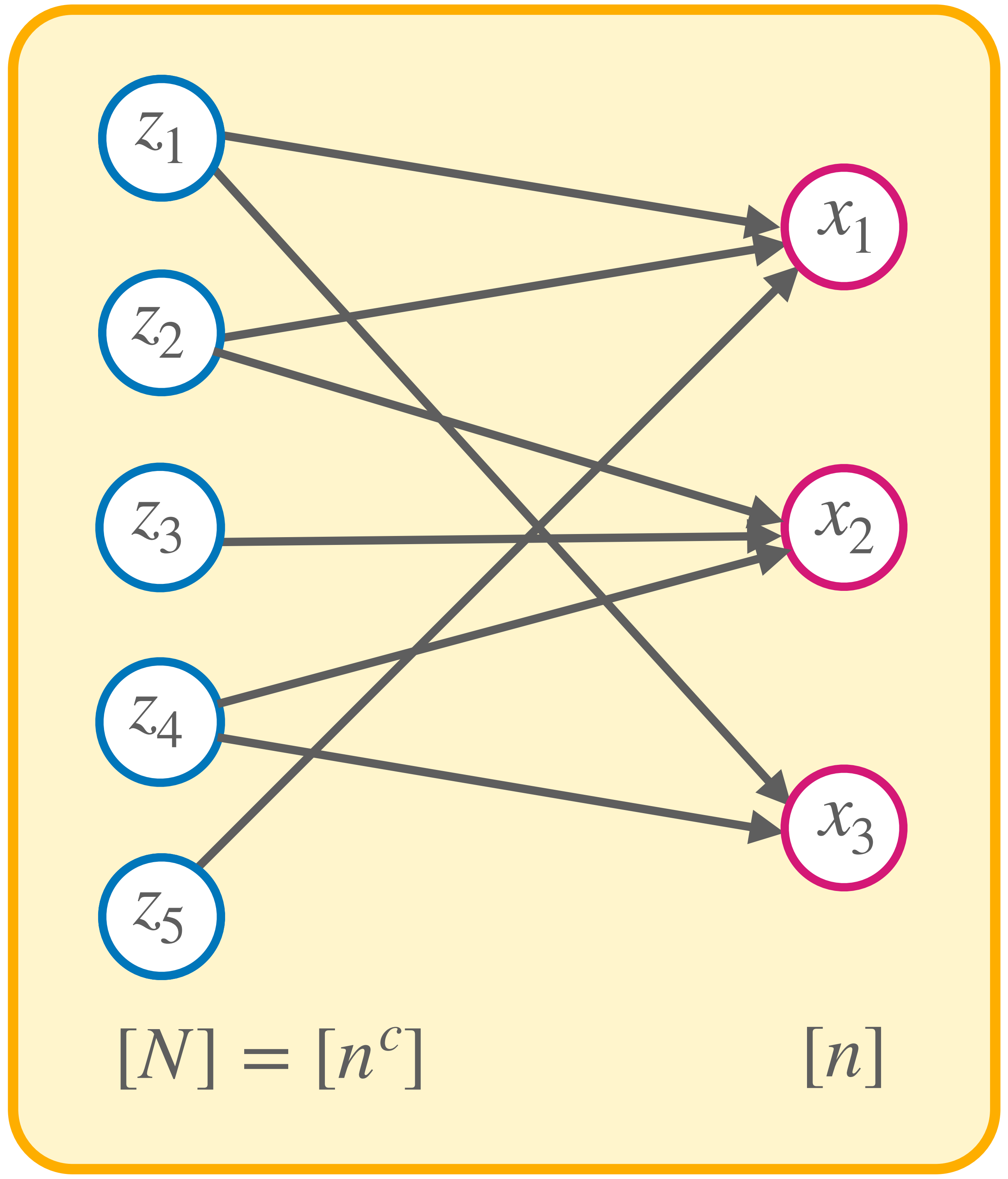

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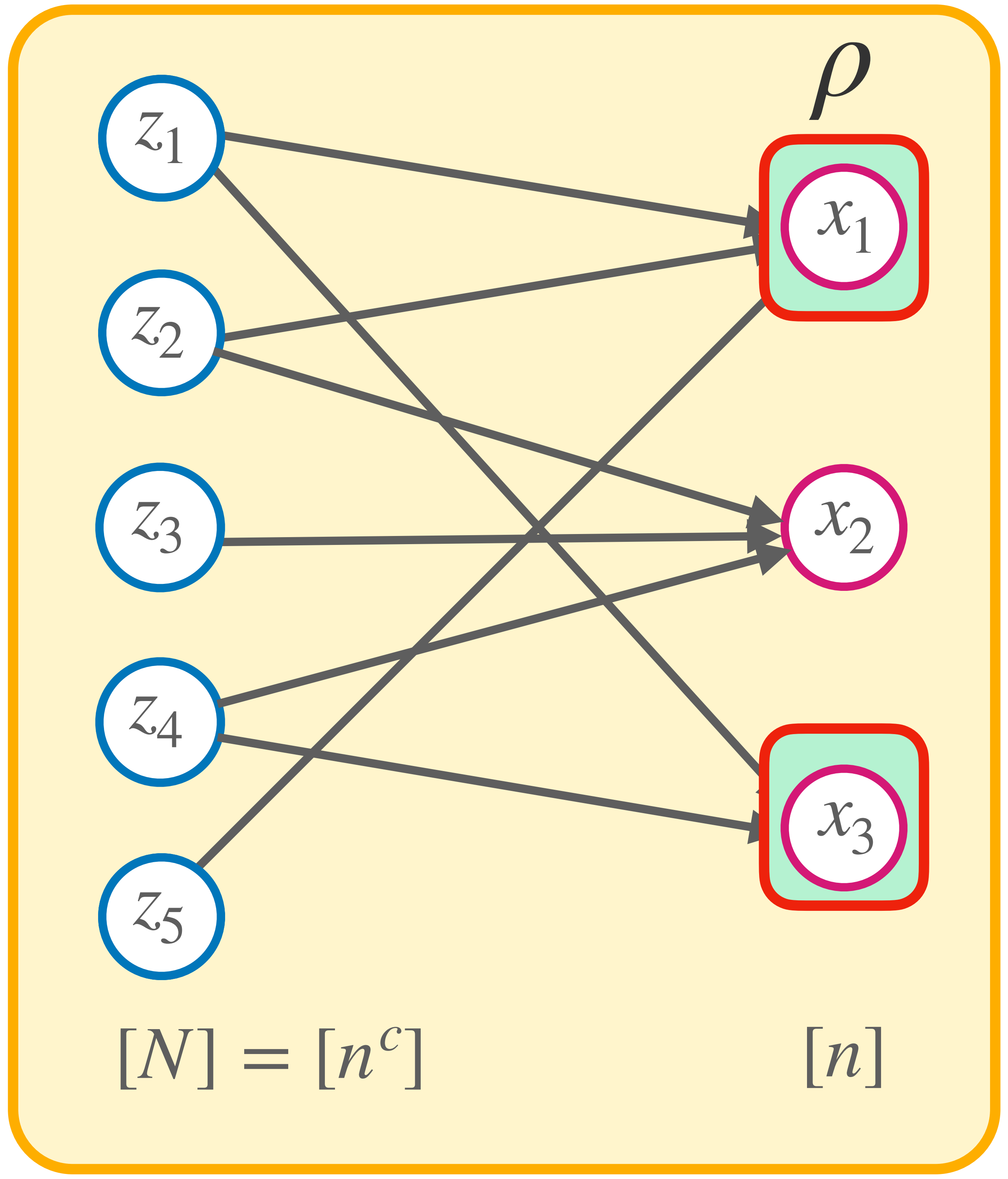

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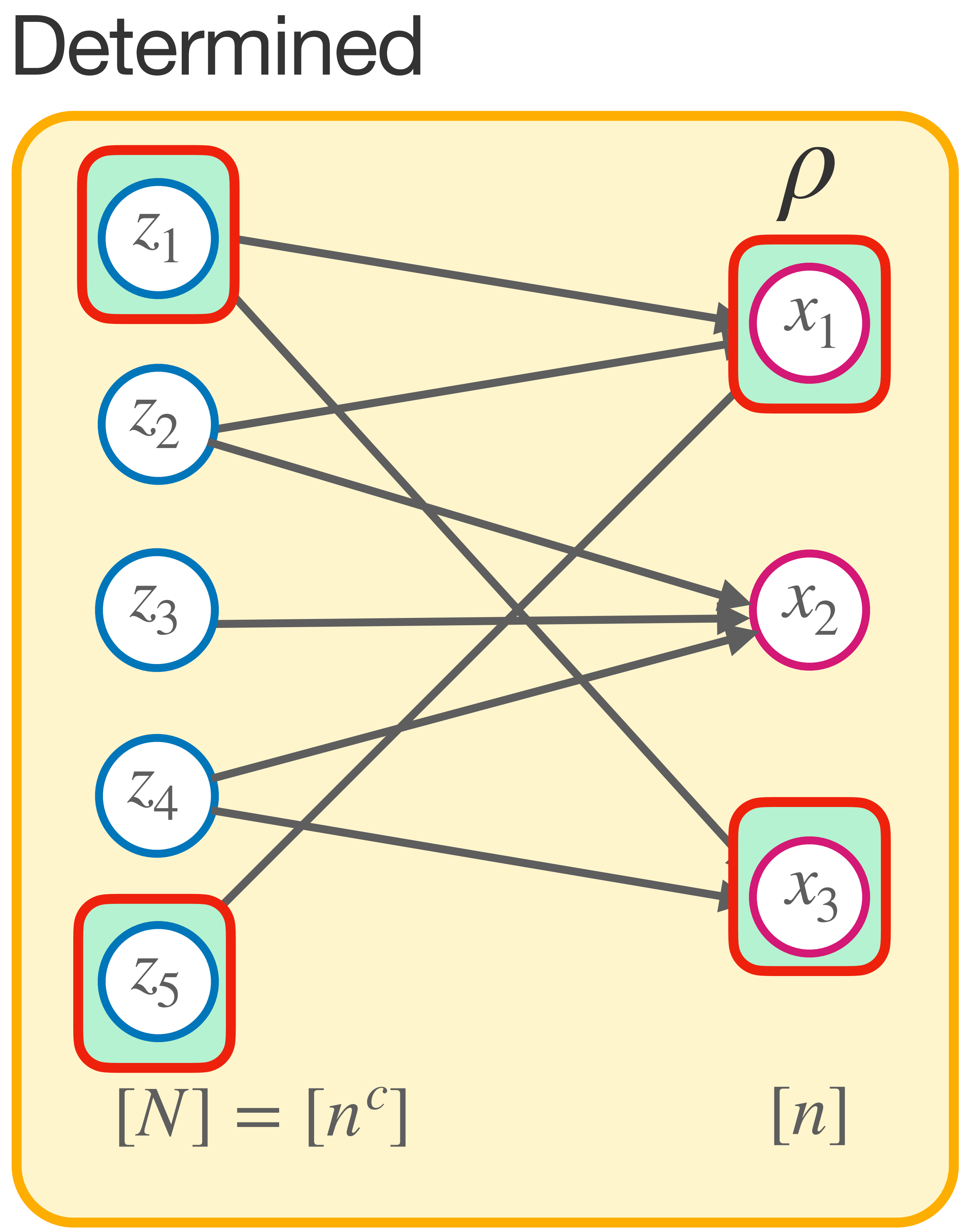

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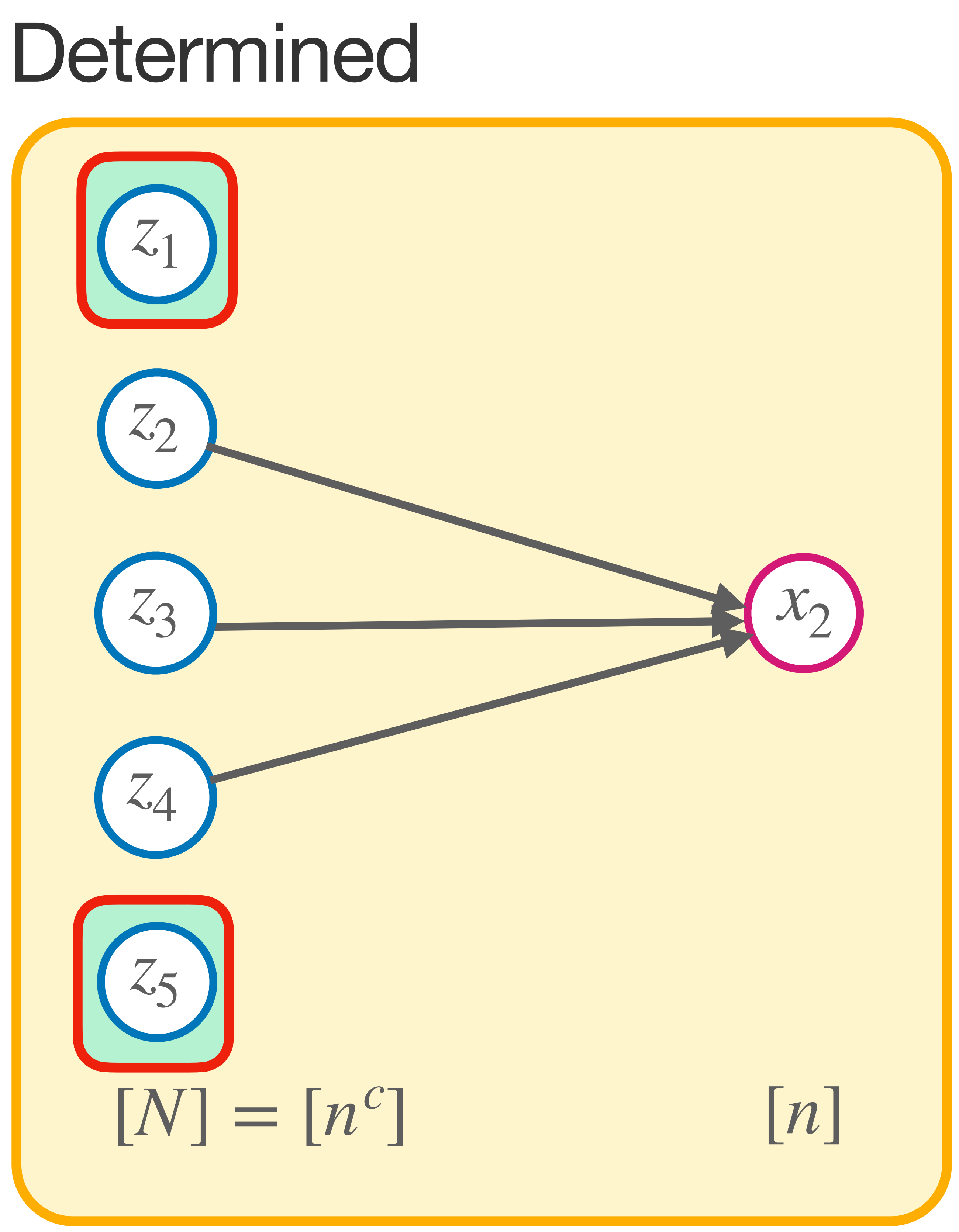

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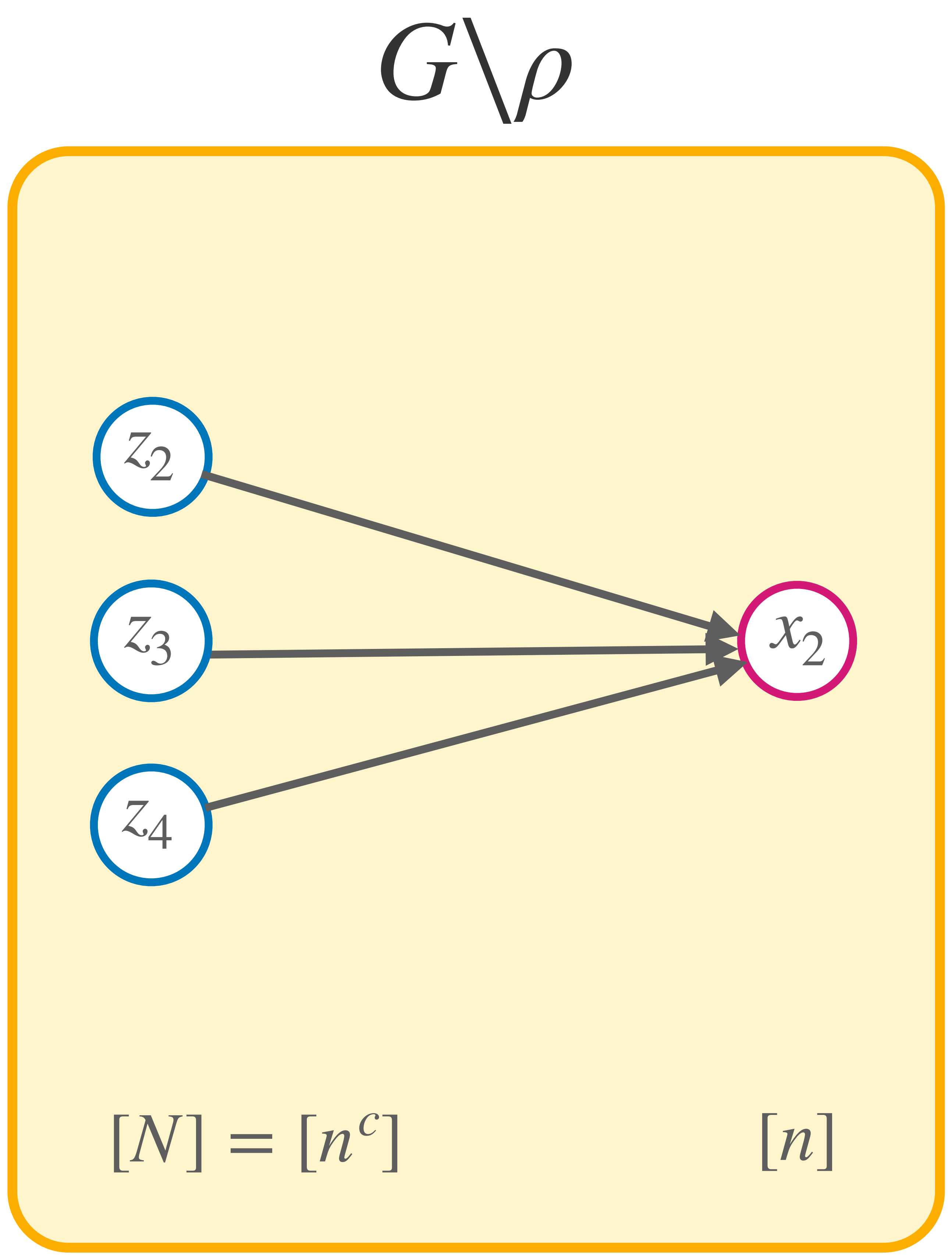

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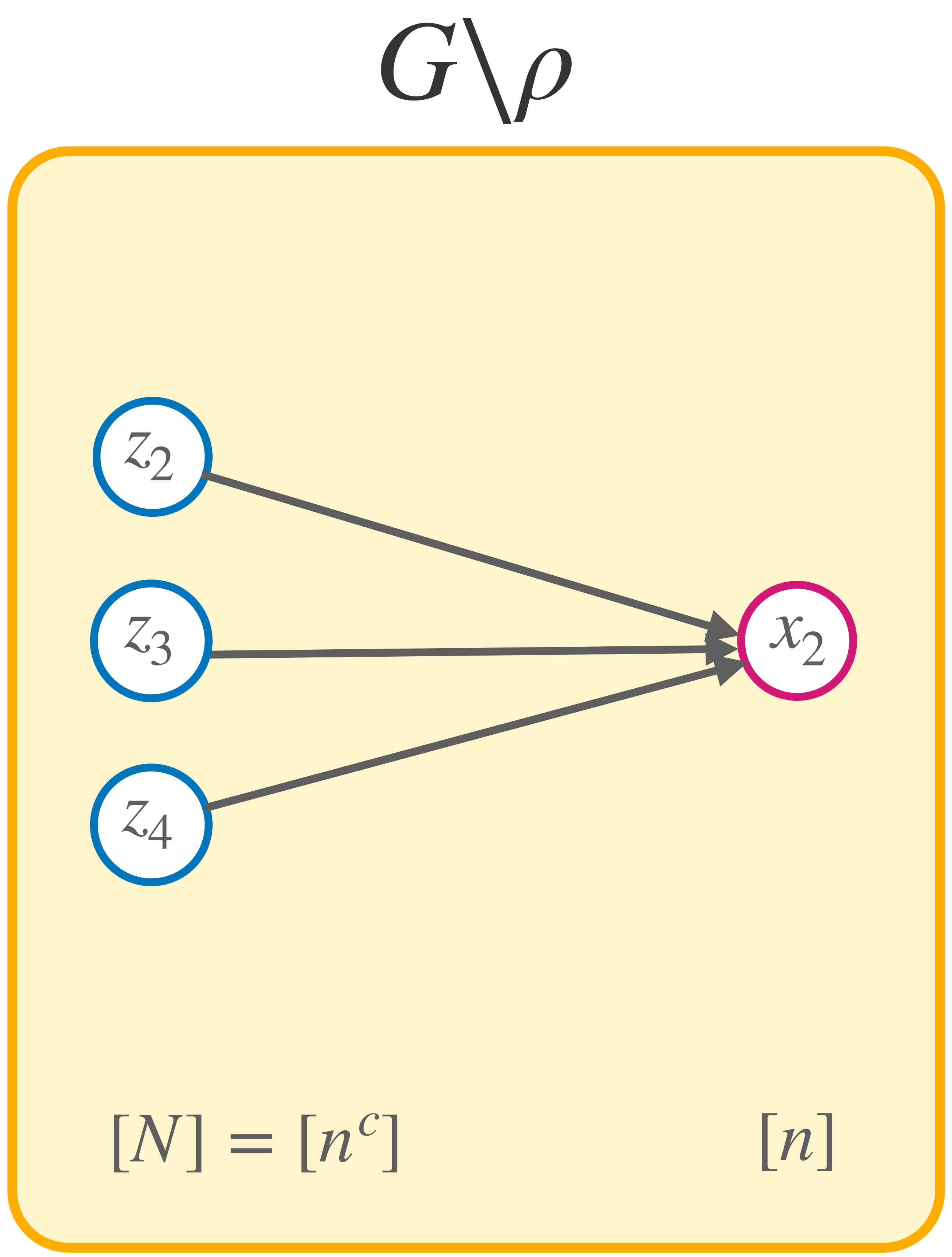

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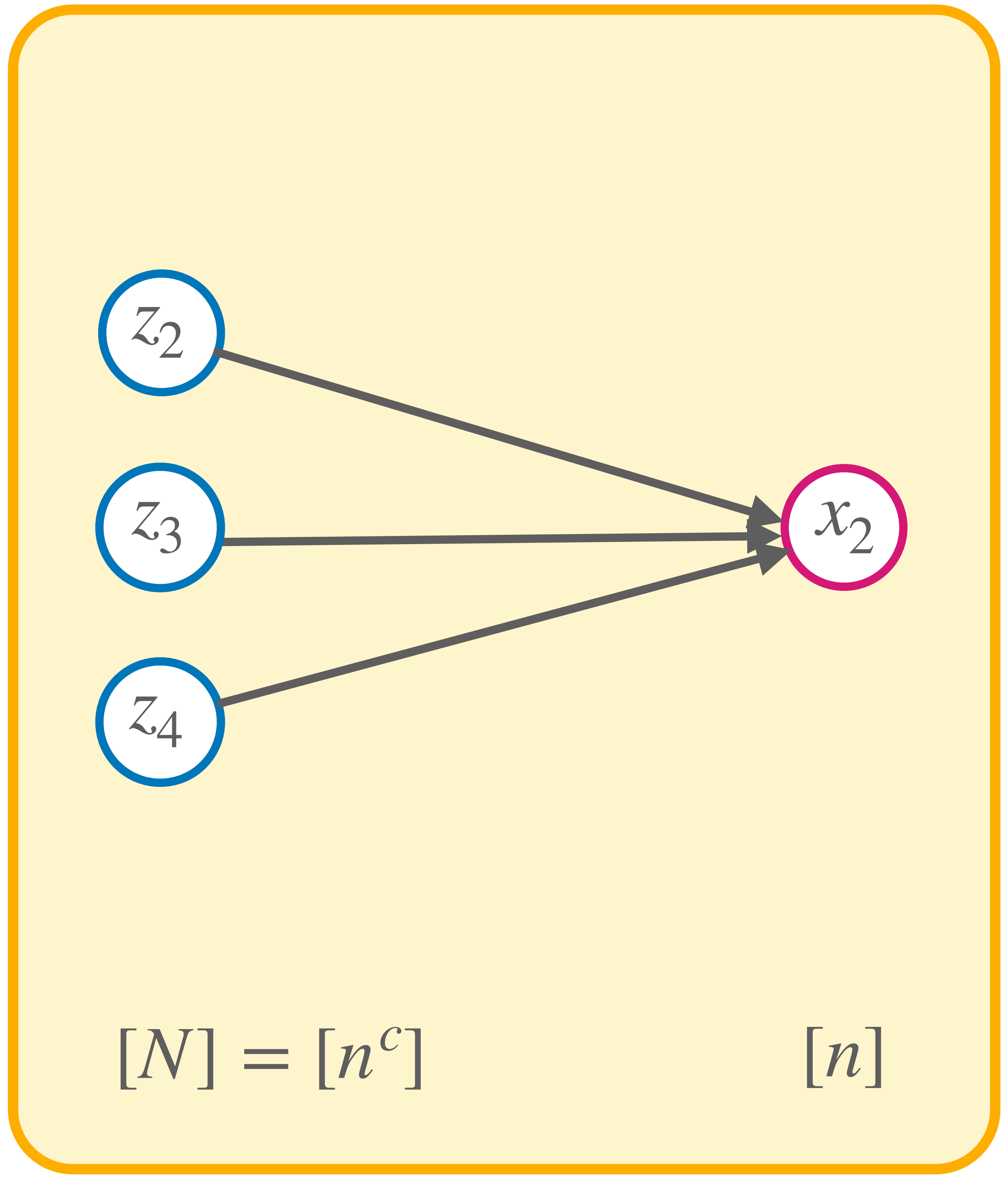

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→ To restore expansion, set the variables of $\mathrm{Cl}(\rho')\backslash\mathtt{vars}(\rho')$!

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Each round uses $O(w)$ queries to $A \implies$ we can continue for $\Omega(d/w)$ rounds!

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Interpolation: Any Cutting Planes proof of any F implies a monotone circuits computing an associated function f_F with the same topology [P96, HP17, FPPR17]

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One approach…

Can the Ben-Sasson Wigderson size-width relation be balanced?

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Q. Supercritical size/depth tradeoffs for non-monotone circuits?